Notes on Classical Mechanics

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1. These are notes prepared for the benefit of students enrolled in Classical Mechanics (PHYS-510) at Southern Illinois University–Carbondale. It will be updated periodically, and will evolve during the semester. It is not a substitute for standard textbooks, but a supplement prepared as a study-guide.

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Contents

Chapter 1

Newton's laws of motion

1.1 Position dependent forces

1. (**20 points.**) Radial free fall of a meteoroid. Refer 20210121 video.
2. (**20 points.**) (Refer Landau and Lifshitz, Problem 1 in Chapter 3.) A simple pendulum consists of a particle of mass m suspended by a massless rod of length l in a uniform gravitational field g.
   (a) Identify the two forces acting on the pendulum to be the force of gravity $m\mathbf{g}$ and the force of tension $\mathbf{T}$. Thus, deduce the Newton equation of motion to be
   $$m\mathbf{a} = m\mathbf{g} + \mathbf{T}, \tag{1.1}$$
   where $\mathbf{a}$ is acceleration of mass m. Starting from Eq. (1.1) derive the equation of motion for the simple pendulum
   $$\frac{d^2\phi}{dt^2} = -\omega_0^2 \sin\phi, \tag{1.2}$$
   where
   $$\omega_0 = \frac{2\pi}{T_0} = \sqrt{\frac{g}{l}}. \tag{1.3}$$
   (b) Starting from Eq. (1.2) derive the statement of conservation of energy for this system,
   $$\frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi = \text{constant}. \tag{1.4}$$
   Hint: Multiply Eq. (1.2) by $\dot{\phi}$ and express the equation as a total derivative with respect to time.
   (c) For initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$ show that
   $$\frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi = -mgl\cos\phi_0. \tag{1.5}$$
   Thus, derive
   $$\frac{dt}{T_0} = \frac{1}{2\pi}\frac{d\phi}{\sqrt{2(\cos\phi - \cos\phi_0)}} \tag{1.6}$$
   where $T_0 = 2\pi\sqrt{l/g}$.
   (d) The time period of oscillations of the simple pendulum is equal to four times the time taken between $\phi = 0$ and $\phi = \phi_0$. Thus, show that
   $$T = 4\frac{T_0}{2\pi}\int_0^{\phi_0}\frac{d\phi}{\sqrt{2(\cos\phi - \cos\phi_0)}} \tag{1.7}$$
   $$= \frac{T_0}{\pi}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}}}. \tag{1.8}$$

Then, substitute $\sin\theta = \sin(\phi/2)/\sin(\phi_0/2)$ to determine the period of oscillations of the simple pendulum as a function of the amplitude of oscillations ϕ_0 to be

$$T = T_0 \frac{2}{\pi} K\left(\sin\frac{\phi_0}{2}\right), \tag{1.9}$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{1.10}$$

is the complete elliptic integral of the first kind.

(e) Using the power series expansion

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} \tag{1.11}$$

show that for small oscillations ($\phi_0/2 \ll 1$)

$$T = T_0\left[1 + \frac{\phi_0^2}{16} + \ldots\right]. \tag{1.12}$$

(f) Estimate the percentage error made in the approximation $T \sim T_0$ for $\phi_0 \sim 60°$.

(g) Plot the time period T of Eq. (1.9) as a function of ϕ_0. What can you conclude about the time period for $\phi_0 = \pi$?

3. (**20 points.**) Assume Earth to be a solid spherical ball of uniform density. Consider a hypothetical tunnel passing through the center of Earth and connecting two points on the surface of Earth by a straight line. Determine the time taken, (in minutes) to two siginificant digits, starting from rest, to travel from one point to the other, when a mass is dropped at one end of the tunnel. Ignore friction and the rotational motion of Earth. Use the mass of Earth to be 6.0×10^{24} kg, radius of Earth to be 6.4×10^6 m. Newton's gravitational constant is $6.67 \times 10^{-11}\,\mathrm{Nm^2/kg^2}$.

A more realistic density profile of Earth is

$$\rho(r) = \begin{cases} \rho_0, & \text{for} \quad r < \frac{R}{2}, \\ \frac{1}{2}\rho_0, & \text{for} \quad \frac{R}{2} < r < R, \end{cases} \tag{1.13}$$

where

$$\rho_0 = \frac{16}{9}\frac{M}{\frac{4\pi}{3}R^3}, \tag{1.14}$$

where R is the radius of Earth and M is the mass of Earth. Show that the above density profile leads to the following profile for the gravitational field for Earth,

$$g(r) = \begin{cases} -\dfrac{16}{9}\dfrac{GMr}{R^3}, & \text{for} \quad r < \dfrac{R}{2}, \\[2ex] -\dfrac{8}{9}\dfrac{GM}{R^2}\dfrac{1}{2}\left[\dfrac{2r}{R} + \left(\dfrac{R}{2r}\right)^2\right], & \text{for} \quad \dfrac{R}{2} < r < R, \\[2ex] -\dfrac{GM}{r^2}, & \text{for} \quad R < r, \end{cases} \tag{1.15}$$

where G is Newton's gravitational constant. Plot $g(r)$ as a function of r. Approximate the above gravitational field as

$$g(r) \approx \begin{cases} -\dfrac{GM}{R^2}\dfrac{2r}{R}, & \text{for} \quad r < \dfrac{R}{2}, \\ -\dfrac{GM}{R^2}, & \text{for} \quad \dfrac{R}{2} < r < R, \\ -\dfrac{GM}{r^2}, & \text{for} \quad R < r. \end{cases} \tag{1.16}$$

Plot the approximate gravitational field and compare it with the exact version. Argue that it is accurate to about ten percent. Determine the new time taken, (in minutes) to two siginificant digits, starting from rest, to travel from one point to the other, when a mass is dropped at one end of the tunnel. Ignore friction and the rotational motion of Earth.
Refer: The gravity tunnel in a non-uniform Earth, by Alexander R. Klotz, Am. J. Phys. 83 (2015) 231; arXiv:1308.1342.

1.2 Velocity dependent forces

1. (**20 points.**) Consider the case when the friction force is quadratically proportional to velocity,

$$F_f = \frac{1}{2} D\rho A v^2, \tag{1.17}$$

where A is the area of crosssection of the object, ρ is the density of the medium, and D is a dimensionless drag coefficient. This should be contrasted with the case when the drag is linear in velocity. Typically, for small speeds, or when the size of the object is small, the drag force is linear in velocity. This is the case for motion in a highly viscous fluid, or for micron sized organisms in water. On the other hand, a sky diver, or a car on an interstate, experience quadratic drag forces.

(a) For a mass m falling under uniform gravity we have the equation of motion

$$m\frac{dv}{dt} = mg - F_f. \tag{1.18}$$

(b) Show that the terminal velocity, when $dv/dt = 0$, is given by

$$v_T = \sqrt{\frac{2mg}{D\rho A}}. \tag{1.19}$$

(c) Solve the equation of motion for the initial condition where the particle starts from rest, $v(0) = 0$, and show that it leads to the solution

$$v(t) = v_T \frac{\left(1 - e^{-\frac{2t}{\tau}}\right)}{\left(1 + e^{-\frac{2t}{\tau}}\right)}, \tag{1.20}$$

where $\tau = v_T/g$ sets the scale for time.

(d) The corresponding solution for linear drag is

$$v(t) = v_T\left(1 - e^{-\frac{t}{\tau}}\right), \tag{1.21}$$

where now $F_f = bv$ and $v_T = \frac{mg}{b}$ with $\tau = v_T/g$. Plot and compare the two velocity functions assuming the same terminal velocities.

2. (**20 points.**) Electric charge in a uniform magnetic field. Refer 20210121 video.

3. (**20 points.**) Motion of a charged particle of mass m and charge q in a uniform magnetic field $\mathbf{B}$ and a uniform electric field $\mathbf{E}$ is governed by

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{E} + q\,\mathbf{v} \times \mathbf{B}. \tag{1.22}$$

Choose $\mathbf{B}$ along the z-axis and $\mathbf{E}$ along the y-axis,

$$\mathbf{B} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + B\,\hat{\mathbf{k}}, \tag{1.23a}$$

$$\mathbf{E} = 0\,\hat{\mathbf{i}} + E\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{1.23b}$$

Solve this vector differential equation to determine the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ of the particle as a function of time, for initial conditions

$$\mathbf{x}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{1.24a}$$

$$\mathbf{v}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{1.24b}$$

Verify that the solution is a cycloid characterized by the equations

$$x(t) = R(\omega_c t - \sin\omega_c t), \tag{1.25a}$$

$$y(t) = R(1 - \cos\omega_c t). \tag{1.25b}$$

where

$$R = \frac{E}{B\omega_c}, \qquad \omega_c = \frac{qB}{m}. \tag{1.26}$$

The particle moves as though it were a point on the rim of a wheel of radius R perfectly rolling (without sliding or slipping) with angular speed ω_c along the x-axis. It satisfies the equation of a circle of radius R whose center $(vt, R, 0)$ travels along the x-direction at constant speed v,

$$(x - vt)^2 + (y - R)^2 = R^2, \tag{1.27}$$

where $v = \omega_c R$.

Chapter 2

Calculus of variations

2.1 Functional derivative

1. (**20 points.**) Give an account of the functional derivative

$$\frac{\delta u(x)}{\delta u(x')} = \delta(x - x'). \tag{2.1}$$

Observe that dimensional consistency requires

$$\left[\frac{\delta}{\delta u(x)}\right] = \frac{1}{[u][x]}. \tag{2.2}$$

2. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)} \tag{2.3}$$

of the following functionals, assuming no variation at the end points.

(a)

$$F[u] = \int_{x_1}^{x_2} dx\, a(x) u(x) \tag{2.4}$$

(b)

$$F[u] = \int_{x_1}^{x_2} dx\, u(x)^2 \tag{2.5}$$

(c)

$$F[u] = \int_{x_1}^{x_2} dx\, a(x) \frac{du(x)}{dx} \tag{2.6}$$

3. (**20 points.**) (Gelfand and Fomin, Calculus of Variations.) Evaluate the functional derivative

$$\frac{\delta F[y]}{\delta y(x)} \tag{2.7}$$

of the following functionals, assuming no variation at the end points.

(a)

$$F[y] = \int_0^1 dx\, \frac{dy}{dx} \tag{2.8}$$

(b)

$$F[y] = \int_0^1 dx\, y \frac{dy}{dx} \tag{2.9}$$

(c)

$$F[y] = \int_0^1 dx\, xy \frac{dy}{dx} \tag{2.10}$$

(d)

$$F[y] = \int_a^b \frac{dx}{x^3} \left(\frac{dy}{dx}\right)^2 \tag{2.11}$$

4. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)} \tag{2.12}$$

of the following functionals, assuming no variation at the end points. Given $a(x)$ is a known function.

(a)

$$F[u] = \int_{x_1}^{x_2} dx\, a(x) \left[1 + \frac{du(x)}{dx} + \frac{d^2u(x)}{dx^2} + \frac{d^3u(x)}{dx^3}\right] \tag{2.13}$$

(b)

$$F[u] = \int_a^b dx \frac{1}{\left(1 + \frac{d^3u}{dx^3}\right)} \tag{2.14}$$

(c)

$$F[u] = \int_a^b dx\, x^5 \sqrt{1 + \frac{d^3u}{dx^3}} \tag{2.15}$$

(d)

$$F[u] = \int_a^b dx \sqrt{1 + \frac{du}{dx} + \frac{d^3u}{dx^3}} \tag{2.16}$$

5. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta W[x]}{\delta x(t)} \tag{2.17}$$

of the following functionals, assuming no variation at the end points.

(a) Let $x(t)$ be position at time t of mass m. The action

$$W[x] = \int_{t_1}^{t_2} dt\, \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \tag{2.18}$$

is a functional of position.

(b) Let $z(t)$ be the vertical height at time t of mass m in a uniform gravitational field g. The action

$$W[z] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dz}{dt}\right)^2 - mgz\right] \tag{2.19}$$

is a functional of the vertical height.

(c) Let $r(t)$ be the radial distance at time t of mass m released from rest in a gravitational field of a planet of mass M. The action

$$W[r] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{GMm}{r} \right] \tag{2.20}$$

is a functional of the radial distance.

(d) Let $r(t)$ be the radial distance at time t of charge q_1 of mass m released from rest in an electrostatic field of another charge of charge q_2. The action

$$W[r] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dr}{dt} \right)^2 - \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r} \right] \tag{2.21}$$

is a functional of the radial distance.

6. (**20 points.**) Let us investigate the fundamental identity of functional differentiation,

$$\frac{\delta f(x)}{\delta f(x')} = \delta(x - x'), \tag{2.22}$$

in the context of Fourier tranformation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k), \tag{2.23a}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \, e^{-ikx} f(x). \tag{2.23b}$$

Observe that the above Fourier transformation implies the δ-function representation

$$\delta(x - x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')}. \tag{2.24}$$

Interpreting Eq. (2.23a) as a functional in $\tilde{f}$ show that

$$\frac{\delta f(x)}{\delta \tilde{f}(k)} = \frac{1}{2\pi} e^{ikx}. \tag{2.25}$$

Similarly, interpreting Eq. (2.23b) as a functional in f show that

$$\frac{\delta \tilde{f}(k)}{\delta f(x)} = e^{-ikx}. \tag{2.26}$$

Using these results in the functional chain rule

$$\frac{\delta f(x)}{\delta f(x')} = \int_{-\infty}^{\infty} dk \, \frac{\delta f(x)}{\delta \tilde{f}(k)} \frac{\delta \tilde{f}(k)}{\delta f(x')} \tag{2.27}$$

obtain the fundamental identity in Eq. (2.22).

7. (**20 points.**) The eletrostatic energy of a charge distribution $\rho(\mathbf{r})$ is

$$E[\rho] = \frac{1}{2} \int d^3r \int d^3r' \, \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{2.28}$$

Evaluate

$$\frac{\delta^2 E}{\delta\rho(\mathbf{r})\rho(\mathbf{r}')}. \tag{2.29}$$

8. (**20 points.**) Consider the action, in terms of the Lagrangian viewpoint,

$$W[\mathbf{x}] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{d\mathbf{x}}{dt} \right)^2 - U(\mathbf{x}, t) \right]. \tag{2.30}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivative

$$\frac{\delta W}{\delta \mathbf{x}(t)}. \tag{2.31}$$

9. (**20 points.**) Consider the action, in terms of the Hamiltonian viewpoint,

$$W[\mathbf{x}, \mathbf{p}] = \int_{t_1}^{t_2} dt \left[\mathbf{p} \cdot \frac{d\mathbf{x}}{dt} - \frac{p^2}{2m} - U(\mathbf{x}, t) \right]. \tag{2.32}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivatives

$$\frac{\delta W}{\delta \mathbf{x}(t)} \quad \text{and} \quad \frac{\delta W}{\delta \mathbf{p}(t)}. \tag{2.33}$$

10. (**20 points.**) Consider the action, in terms of the Schwingerian viewpoint,

$$W[\mathbf{x}, \mathbf{p}, \mathbf{v}] = \int_{t_1}^{t_2} dt \left[\mathbf{p} \cdot \left(\frac{d\mathbf{x}}{dt} - \mathbf{v} \right) + \frac{1}{2} m v^2 - U(\mathbf{x}, t) \right]. \tag{2.34}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivatives

$$\frac{\delta W}{\delta \mathbf{x}(t)}, \quad \frac{\delta W}{\delta \mathbf{v}(t)}, \quad \text{and} \quad \frac{\delta W}{\delta \mathbf{p}(t)}. \tag{2.35}$$

11. (**40 points.**) Consider the following construction in a field theoretical setup

$$W[K] = \frac{1}{2} \int dx \int dx' K(x) \Delta(|x - x'|) K(x'), \tag{2.36}$$

where W is the action written in terms of a source function $K(x)$ and the Green's function $\Delta(|x - x'|)$. Determine the relation between the corresponding field $\phi(x)$ and the source, by evaluating the functional derivative

$$\phi(x) = \frac{\delta W}{\delta K(x)}. \tag{2.37}$$

Show that the Green's function satisfies

$$\Delta(|x - x'|) = \frac{\delta^2 W}{\delta K(x) \delta K(x')}. \tag{2.38}$$

Construct the partition function

$$Z[K] = e^{iW[K]}. \tag{2.39}$$

Show that

(a) the field satisfies

$$\phi(x) = \frac{1}{i} \frac{\delta \ln Z}{\delta K(x)} \tag{2.40}$$

(b) and the Green's function is given by

$$\Delta(|x - x'|) = \frac{1}{i} \frac{1}{Z} \frac{\delta^2 Z}{\delta K(x) \delta K(x')} \bigg|_{K=0}. \tag{2.41}$$

12. (**40 points.**) Consider the functional

$$W[x] = \int_{t_1}^{t_2} dt\, L(x, \dot{x}) \tag{2.42}$$

constructed out of the function $x = x(t)$ and its derivative $\dot{x} = dx/dt$. In particular, let

$$\frac{\partial L}{\partial t} = 0. \tag{2.43}$$

(a) Show that

$$\frac{\delta I[x]}{\delta x(t)} = \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] + \Big[\delta(t - t_2) - \delta(t - t_1)\Big]\frac{\partial L}{\partial \dot{x}}. \tag{2.44}$$

(b) Further, show that

$$\frac{\delta I[x]}{\delta x(t)} = \frac{1}{\dot{x}}\frac{d}{dt}\left(L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right) + \Big[\delta(t - t_2) - \delta(t - t_1)\Big]\frac{\partial L}{\partial \dot{x}}. \tag{2.45}$$

This property used with the extremum principle, is the essence of the Beltrami identity. This also gives us a glimpse of the Legendre transform,

$$H = \dot{x}\frac{\partial L}{\partial \dot{x}} - L. \tag{2.46}$$

2.2 Fermat's principle

1. (**20 points.**) Fermat's principle in ray optics states that a ray of light takes the path of least time between two given points. The speed of light in a medium is given in terms of the refractive index

$$n = \frac{c}{v}, \tag{2.47}$$

of the medium, where c is the speed of light in vacuum and v is the speed of light in the medium. Consider a ray of light traversing a path from (x_1, y_1) to (x_2, y_2) in a plane of fixed z.

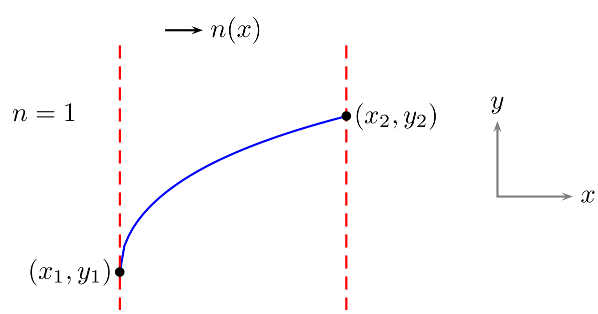

Figure 2.1: Problem 1.

(a) Show that the time taken to travel an infinitesimal distance ds is given by

$$dt = \frac{ds}{v} = \frac{n\, ds}{c}, \tag{2.48}$$

where ds in a plane is characterized by the infinitesimal statement

$$ds^2 = dx^2 + dy^2. \tag{2.49}$$

(b) Fermat's principle states that the path traversed by a ray of light from (x_1, y_1) to (x_2, y_2) is the extremal of the functional

$$T[y] = \frac{1}{c} \int_{(x_1,y_1)}^{(x_2,y_2)} n\, ds = \frac{1}{c} \int_{x_1}^{x_2} dx\, n(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{2.50}$$

(c) Since the ray of light passes through the points (x_1, y_1) and (x_2, y_2), we do not consider variations at these (end) points. Thus, show that

$$\frac{\delta T[y]}{\delta y(x)} = -\frac{1}{c}\frac{d}{dx}\left[\frac{n(x)\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right]. \tag{2.51}$$

(d) Using Fermat's principle show that the differential equation for the path $y(x)$ traversed by the ray of light is

$$\frac{n(x)\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = n_0, \tag{2.52}$$

where n_0 is a constant. Show that the above equation can be rewritten in the form

$$\frac{dy}{dx} = \frac{n_0}{\sqrt{n(x)^2 - n_0^2}}. \tag{2.53}$$

(e) Let us consider a medium with refractive index $(x_1 = a)$

$$n(x) = \begin{cases} 1, & x < a, \\ \frac{x}{a}, & a < x. \end{cases} \tag{2.54}$$

Solve the corresponding differential equation, by substituting $x = n_0 a \cosh t$, to obtain

$$y(x) - y_0 = n_0 a \cosh^{-1}\left(\frac{1}{n_0}\frac{x}{a}\right), \qquad a < x. \tag{2.55}$$

The path in this medium satisfies the equation of a catenary. It is also useful to express the solution in terms of the logarithm as

$$y(x) - y_0 = n_0 a \ln\left[\frac{1}{n_0}\frac{x}{a} + \sqrt{\left(\frac{1}{n_0}\frac{x}{a}\right)^2 - 1}\right], \qquad a < x. \tag{2.56}$$

For initial conditions

$$y(x_1) = y_1 \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{x=x_1} = y_1' \tag{2.57}$$

show that integration constants are determined as

$$y_0 = y_1 \qquad \text{and} \qquad n_0 = \frac{y_1'}{\sqrt{1 + {y_1'}^2}}. \tag{2.58}$$

Thus, write the solution as

$$y(x) - y_1 = n_0 a \ln \left[\frac{\frac{1}{n_0}\frac{x}{a} + \sqrt{\frac{1}{n_0^2}\frac{x^2}{a^2} - 1}}{\frac{1}{n_0} + \sqrt{\frac{1}{n_0^2} - 1}} \right], \qquad a < x. \tag{2.59}$$

For the special case $y_1 = 0$ and $y_1' \to \infty$ show that $n_0 = 1$ and

$$y(x) = a \ln \left[\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right], \qquad a < x. \tag{2.60}$$

Evaluate the total time taken for light to go from (x_1, y_1) to $(x, y(x))$.
Solution:

$$T = \frac{a}{c}\frac{n_0^2}{2} \left[\ln \left(\frac{x + \sqrt{x^2 - n_0^2 a^2}}{a + \sqrt{a^2 - n_0^2 a^2}} \right) + \frac{x}{n_0 a}\sqrt{\frac{x^2}{n_0^2 a^2} - 1} - \frac{1}{n_0}\sqrt{\frac{1}{n_0^2} - 1} \right], \qquad a < x. \tag{2.61}$$

2. (**20 points.**) Fermat's principle in ray optics states that a ray of light takes the path of least time between two given points. The speed of light in a medium is given in terms of the refractive index

$$n = \frac{c}{v}, \tag{2.62}$$

of the medium, where c is the speed of light in vacuum and v is the speed of light in the medium. Consider a ray of light traversing a path from (x_1, y_1) to (x_2, y_2) in a plane of fixed z.

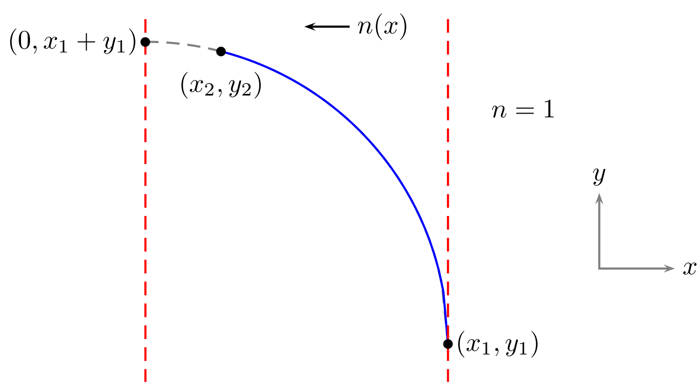

Figure 2.2: Problem 2.

(a) Show that the time taken to travel an infinitesimal distance ds is given by

$$dt = \frac{ds}{v} = \frac{n\, ds}{c}, \tag{2.63}$$

where ds in a plane is characterized by the infinitesimal statement

$$ds^2 = dx^2 + dy^2. \tag{2.64}$$

(b) Fermat's principle states that the path traversed by a ray of light from (x_1, y_1) to (x_2, y_2) is the extremal of the functional

$$T[y] = \frac{1}{c}\int_{(x_1,y_1)}^{(x_2,y_2)} n\, ds = \frac{1}{c}\int_{x_1}^{x_2} dx\, n(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{2.65}$$

(c) Since the ray of light passes through the points (x_1, y_1) and (x_2, y_2), we do not consider variations at these (end) points. Thus, show that

$$\frac{\delta T[y]}{\delta y(x)} = -\frac{1}{c}\frac{d}{dx}\left[\frac{n(x)\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}\right]. \tag{2.66}$$

(d) Using Fermat's principle show that the differential equation for the path $y(x)$ traversed by the ray of light is

$$\frac{n(x)\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} = n_0, \tag{2.67}$$

where n_0 is a constant. Show that the above equation can be rewritten in the form

$$\frac{dy}{dx} = \frac{n_0}{\sqrt{n(x)^2 - n_0^2}}. \tag{2.68}$$

(e) Let us consider a medium with refractive index $(x_1 = a)$

$$n(x) = \begin{cases} \dfrac{a}{x}, & 0 < x < a, \\ 1, & a < x. \end{cases} \tag{2.69}$$

Solve the corresponding differential equation to obtain

$$y(x) - y_0 = \frac{1}{n_0}\left[\sqrt{a^2 - n_0^2x^2} - \sqrt{a^2 - n_0^2a^2}\right], \qquad x < a. \tag{2.70}$$

The path in this medium satisfies the equation of a circle. Determine the radius of the circle to be a/n_0 and the location of the center to be $(0, y_0 - a\sqrt{(1/n_0^2) - 1})$. For initial conditions

$$y(x_1) = y_1 \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{x=x_1} = y_1' \tag{2.71}$$

show that the integration constants are determined to be

$$y_0 = y_1 \qquad \text{and} \qquad n_0 = \frac{y_1'}{\sqrt{1 + {y_1'}^2}}. \tag{2.72}$$

For the special case when $y_1 = 0$ and $y_1' \to \infty$ show that $n_0 = 1$ and

$$y(x) = \sqrt{a^2 - x^2}, \qquad x < a. \tag{2.73}$$

Evaluate the total time taken for light to go from $(x_1 = a, y_1 = 0)$ to $(x_2 = 0, y_2 = a)$.

(f) **To do:** Check for a sign in the solution for y. Further, should $y_1' \to -\infty$? Refer solutions to MT-01 in Spring 2020.
Solution:
The time taken is given by

$$cT = -\int_a^0 dx\, n(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2} = a\lim_{\delta\to 0}\int_\delta^1 \frac{dt}{t}\frac{1}{\sqrt{1-t^2}}. \tag{2.74}$$

The negative sign in the expression corresponds to velocity being negative. This yields

$$cT = -a \ln \left(\frac{\delta}{1 + \sqrt{1 - \delta^2}} \right) \sim a(\ln 2 - \ln \delta), \tag{2.75}$$

which diverges logarithmically. Thus, the light takes infinite time to reach the point $(0, a)$ from $(a, 0)$.

2.3 Geodesics on surfaces

1. (**20 points.**) Let us prove the intuitively obvious statement that the curve of shortest distance going through two points (x_1, y_1) and (x_2, y_2) in a plane, the geodesics of a plane, is a straight line passing through the two points.

   (a) The distance between two points in a plane is characterized by the infinitesimal statement
   $$ds^2 = dx^2 + dy^2. \tag{2.76}$$

   (b) The geodesic is the extremal of the functional
   $$l[y] = \int_{(x_1,y_1)}^{(x_2,y_2)} ds = \int_{x_1}^{x_2} dx \sqrt{1 + \left(\frac{dy}{dx} \right)^2}. \tag{2.77}$$

   (c) Since the curve passes through the points (x_1, y_1) and (x_2, y_2), we have no variations at these (end) points. Thus, show that
   $$\frac{\delta l[y]}{\delta y(x)} = -\frac{d}{dx} \left[\frac{\dfrac{dy}{dx}}{\sqrt{1 + \left(\dfrac{dy}{dx} \right)^2}} \right]. \tag{2.78}$$

   (d) Using the extremum principle show that the differential equation for the geodesic is
   $$\frac{dy}{dx} = c, \tag{2.79}$$
   where c is a contant.

   (e) Solve the differential equation to identify the equation of a straight line in a plane. Find c.

2. (**20 points.**) Find the geodesics on the surface of a circular cylinder. Identify these curves. Hint: To have a visual perception of these geodesics it helps to note that a cylinder can be mapped (or cut open) into a plane.

   (a) The distance between two points on the surface of a cylinder of radius a is characterized by the infinitesimal statement
   $$ds^2 = a^2 d\phi^2 + dz^2. \tag{2.80}$$

   (b) The geodesic is the extremal of the functional
   $$l[z] = \int_{(\phi_1,z_1)}^{(\phi_2,z_2)} ds = \int_{\phi_1}^{\phi_2} a d\phi \sqrt{1 + \left(\frac{1}{a} \frac{dz}{d\phi} \right)^2}. \tag{2.81}$$

   (c) Since the curve passes through the points (z_1, ϕ_1) and (z_2, ϕ_2) we have no variations on the end points. Thus, show that
   $$\frac{\delta l[z]}{\delta z(\phi)} = -\frac{d}{d\phi} \left[\frac{\dfrac{1}{a} \dfrac{dz}{d\phi}}{\sqrt{1 + \left(\dfrac{1}{a} \dfrac{dz}{d\phi} \right)^2}} \right]. \tag{2.82}$$

(d) Using the extremum principle show that the differential equation for the geodesic is

$$\frac{1}{a}\frac{dz}{d\phi} = c, \tag{2.83}$$

where c is a contant.

(e) Solve the differential equation. Identify the curves described by the solutions. Illustrate a particular curve using a diagram. **Solution:** $z = ca\phi + c_2$. Helix.

3. (**20 points.**) Show that the geodesics on a spherical surface are great circles, that is, circles whose centers lie at the center of the sphere.

(a) The distance between two points on the surface of a sphere of radius a is characterized by the infinitesimal statement

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2. \tag{2.84}$$

(b) The geodesic is the extremal of the functional

$$l[\phi] = \int_{(\theta_1,\phi_1)}^{(\theta_2,\phi_2)} ds = \int_{\theta_1}^{\theta_2} a d\theta \sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}. \tag{2.85}$$

(c) Since the curve passes through the points (θ_1, ϕ_1) and (θ_2, ϕ_2) we have no variations on the end points. Thus, show that

$$\frac{1}{a}\frac{\delta l[\phi]}{\delta\phi(\theta)} = -\frac{d}{d\theta}\left[\frac{\sin^2\theta \dfrac{d\phi}{d\theta}}{\sqrt{1 + \sin^2\theta \left(\dfrac{d\phi}{d\theta}\right)^2}}\right]. \tag{2.86}$$

(d) Using the extremum principle show that the differential equation for the geodesic is

$$\frac{d\phi}{d\theta} = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}, \tag{2.87}$$

where c is an arbitrary constant.

(e) Solve the differential equation to obtain the equation of geodesic as

$$\sin(\phi_0 - \phi) = \bar{c}\cot\theta, \tag{2.88}$$

where $\bar{c} = c/\sqrt{1 - c^2}$, and ϕ_0 is a constant of integration.
Hint: Express the right hand side in terms of $\csc\theta$ and $\cot\theta$, then substitute for $\cot\theta$.

(f) Rewrite the equation of the geodesic in the form

$$-\sin\phi_0 \sin\theta\cos\phi + \cos\phi_0 \sin\theta\sin\phi + \bar{c}\cos\theta = 0. \tag{2.89}$$

Interpret this to be an equation of plane passing through the origin. The condition that this plane has to pass through the two given points determines the constants $\bar{c}$ and ϕ_0, which we shall not attempt here.

4. (**20 points.**) Find the geodesics on the surface of a cone with opening angle θ.
Hint: To have a visual perception of these geodesics it helps to note that a cone can be mapped (or cut open) into a plane.
Solution:

$$r(\phi) = \frac{r_0}{\sin(\sin\theta(\phi - \phi_0))}. \tag{2.90}$$

5. (**20 points.**) Find the geodesics on the surface of a circular cylinder.
Solution: Helix.

$$z(\phi) = c_1\phi + c_2. \tag{2.91}$$

2.4 Brachistochrone on surfaces

6. (**60 points.**) Consider a rope of uniform mass density $\lambda = dm/ds$ hanging from two points, (x_1, y_1) and (x_2, y_2), as shown in Figure 2.3. The gravitational potential energy of an infinitely tiny element of this

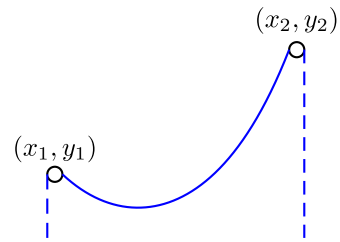

Figure 2.3: Problem 6.

rope at point (x, y) is given by

$$dU = dm\, gy = \lambda g ds\, y, \tag{2.92}$$

where

$$ds^2 = dx^2 + dy^2. \tag{2.93}$$

A catenary is the curve that the rope assumes, that minimizes the total potential energy of the rope.

(a) Show that the total potential energy U of the rope hanging between points x_1 and x_2 is given by

$$U[x] = \lambda g \int_{(x_1,y_1)}^{(x_2,y_2)} y ds = \lambda g \int_{y_1}^{y_2} dy\, y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}. \tag{2.94}$$

(b) Since the curve passes through the points (x_1, y_1) and (x_2, y_2), we have no variations at these (end) points. Thus, show that

$$\frac{\delta U[x]}{\delta x(y)} = -\lambda g \frac{d}{dy} \left[y \frac{\frac{dx}{dy}}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}} \right]. \tag{2.95}$$

(c) Using the extremum principle show that the differential equation for the catenary is

$$\frac{dx}{dy} = \frac{a}{\sqrt{y^2 - a^2}}, \tag{2.96}$$

where a is an integration contant.

(d) Show that integration of the differential equation yields the equation of the catenary

$$y = a \cosh \frac{x - x_0}{a}, \tag{2.97}$$

where x_0 is another integration constant.

(e) For the case $y_1 = y_2$ we have

$$\frac{y_1}{a} = \cosh \frac{x_1 - x_0}{a}, \tag{2.98a}$$

$$\frac{y_2}{a} = \cosh \frac{x_2 - x_0}{a}, \tag{2.98b}$$

which leads to, assuming $x_1 \neq x_2$,

$$x_0 = \frac{x_1 + x_2}{2}. \tag{2.99}$$

Identify x_0 in Figure 2.3. Next, derive

$$\frac{y_1}{a} = \frac{y_2}{a} = \cosh \frac{x_2 - x_1}{2a}, \tag{2.100}$$

which, in principle, determines a. However, this is a transcendental equation in a and does not allow exact evaluation of a, and one depends on numerical solutions. Observe that, if $x = x_0$ in Eq. (2.97), then $y = a$. Identify a in Figure 2.3.

7. (**20 points.**) A catenary is the curve that an idealized hanging chain assumes under its own weight when supported only at its ends in a uniform gravitational field. It is the curve $y(x)$ that minimizes the potential energy U of the hanging chain

$$U = \int dU = \int dm\, gy = \frac{Mg}{P} \int y ds, \tag{2.101}$$

where M is the mass of the uniform chain, P is the length of the chain, g is the acceleration due to gravity. Let us assume the two end points of the chain are at the same height. A catenary is given by

$$y = a \cosh \frac{x}{a}, \tag{2.102}$$

where the parameter a, an integration constant, characterizes the catenary. Find the relation between the parameter a, the perimeter length P of the chain, and the height y_0.

(a) Determine the perimeter length P of the hanging chain using

$$P = \int_{-x_0}^{x_0} ds. \tag{2.103}$$

(b) Show that the relation between the parameter a, the perimeter length P of the chain, and the height y_0 in Figure 2.4 is given by

$$a = \sqrt{y_0^2 - \left(\frac{P}{2}\right)^2} \tag{2.104}$$

(c) Show that the distance x_0 is given by

$$x_0 = a \cosh^{-1} \frac{y_0}{a} = a \ln \left(\frac{y_0}{a} \pm \frac{P}{2a} \right). \tag{2.105}$$

Show that in the limit $a \to 0$ $x_0 \to 0$. This corresponds to the case $y_0 \to P/2$.

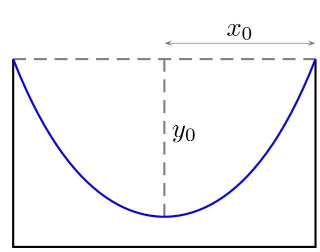

Figure 2.4: Problem 7.

8. (**20 points.**) A catenary is described by

$$y = a \cosh\left(\frac{x - x_0}{a}\right), \tag{2.106}$$

where constants a and x_0 are determined by the position of the end points (x_1, y_1) and (x_2, y_2). Let us choose $x_0 = 0$ and $a = 1$ such that

$$y = \cosh x, \tag{2.107}$$

where x and y are dimensionless variables.

   (a) Using series expansion show that

$$\cosh x = 1 + \frac{x^2}{2} + \ldots. \tag{2.108}$$

   (b) The parabola $y = 1 + x^2/2$ hugs the catenary at $x = 0$, but it does not pass through the end points. Consider the parabola

$$y = 1 + \frac{\alpha}{2} x^2 \tag{2.109}$$

   that also hugs the parabola at $x = 0$. Determine α such that this parabola passes through $x = \pm 1$. Choose this parabola to be an approximation for the catenary. Plot this parabola and the catenary in the same plot for $-1 < x < 1$ and estimate the maximum deviation (with sign) in this approximation. Does the parabola sag below the catenary, or is it the other way around.
   Solution: $\alpha = (\cosh(1) - 1) \sim 1.08616$. Maximum deviation is -0.010 (about 1%). Thus, the parabola sags below the catenary.

9. (**20 points.**) (Based on Problem 7 in Chapter 2 of Goldstein, 2nd edition.) Catenoid: A rope of uniform linear mass density and indefinite length passes freely over pulleys at equal heights $y_1 = y_2$, above the surface of Earth, with horizontal distance $x_2 - x_1$ between them. (Assume uniform gravitational field.) Determine the curve followed by the rope hanging between the pulleys. Compare (using plots) the catenoid and a parabola.

10. (**20 points.**) Write a brief summary on the Isoperimetric problem, and problem of minimum surface of revolution. For example, refer Goldstein, Chapter 2.
   A related note: A gyroid is an infinitely connected triply periodic minimal surface discovered by Alan Schoen, who is a retired faculty of the Math department in SIUC and currently a resident of Carbondale.

Chapter 3

Hamilton's principle

3.1 Euler-Lagrange equation

11. (**20 points.**) (Refer Goldstein, 2nd edition, Chapter 1 Problem 8.) As a consequence of the Hamilton's stationary action principle, the equations of motion for a system can be expressed as Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \tag{3.1}$$

in terms of a Lagrangian $L(x, \dot{x}, t)$. Show that the Lagrangian for a system is not unique. In particular, show that if $L(x, \dot{x}, t)$ satisfies the Euler-Lagrange equation then

$$L'(x, \dot{x}, t) = L(x, \dot{x}, t) + \frac{dF(x, t)}{dt}, \tag{3.2}$$

where $F(x, t)$ is any arbitrary differentiable function, also satisfies the Euler-Lagrange equation.

12. (**20 points.**) A mass m_1 is forced to move on a vertical circle of radius R with uniform angular speed ω. Another mass m_2 is connected to mass m_1 using a massless rod of length a, such that it is a simple pendulum with respect to mass m_1. Motion of both the masses is constrained to be in a vertical plane in a uniform gravitational field.

    (a) Write the Lagrangian for the system.

    (b) Determine the equation of motion for the system.

    (c) Give physical interpretation of each term in the equation of motion.

13. (**20 points.**) A system, characterized by the parameters ω, α, and β, and the dynamical parameter θ, is described by the equation of motion

$$\ddot{\theta} + \omega^2 \sin\theta + \alpha\ddot{\theta}\cos\theta + \beta\dot{\theta}^2 \sin\theta = 0. \tag{3.3}$$

Write the above equation of motion in the small angle approximation, to the leading order in θ.

14. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a horizontal rod. See Figure 14. For simplification, and at loss of generality, let us chose the motion of the pendulum in a vertical plane containing the rod.

    (a) Determine the Lagrangian for the system to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2 a^2\dot{\theta}^2 + m_2 a\dot{x}\dot{\theta}\cos\theta + m_2 g a\cos\theta. \tag{3.4}$$

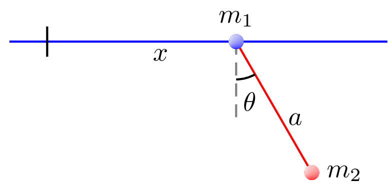

Figure 3.1: Problem 14.

(b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot{x}} = m_2\dot{x} + m_2 a\dot{\theta}\cos\theta, \tag{3.5a}$$
$$\frac{\partial L}{\partial x} = 0, \tag{3.5b}$$
$$\frac{\partial L}{\partial \dot{\theta}} = m_2 a^2\dot{\theta} + m_2 a\dot{x}\cos\theta, \tag{3.5c}$$
$$\frac{\partial L}{\partial \theta} = -m_2 a\dot{x}\dot{\theta}\sin\theta - m_2 g a\sin\theta. \tag{3.5d}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta = 0, \tag{3.6a}$$
$$a\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0. \tag{3.6b}$$

Observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

(d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{x} + a\ddot{\theta} = 0, \tag{3.7a}$$
$$a\ddot{\theta} + \ddot{x} + g\theta = 0. \tag{3.7b}$$

Determine the solution to be given by

$$\theta = 0 \quad \text{and} \quad \ddot{x} = 0. \tag{3.8}$$

Interpret this solution.

(e) The solution $\theta = 0$ seems to be too restrictive. Will this system not allow $\theta \neq 0$? To investigate this, let us not restrict to the small angle approximation. Rewrite Eqs. (3.6), using Eq. (3.6a) in Eq. (3.6b), as

$$\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta = 0, \tag{3.9a}$$
$$\sin\theta\Big[a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta + g\Big] = 0. \tag{3.9b}$$

In this form we immediately observe that $\theta = 0$ is a solution. However, it is not the only solution. Towards interpretting Eqs. (3.9) let us identify the coordinates of the center of mass of the m_1-m_2 system,

$$(m_1 + m_2)x_{\text{cm}} = m_1 x + m_2(x + a\sin\theta), \tag{3.10a}$$
$$(m_1 + m_2)y_{\text{cm}} = -m_2 a\cos\theta, \tag{3.10b}$$

which for $m_1 = 0$ are the coordinates of the mass m_2,

$$x_{\rm cm} = x + a\sin\theta, \tag{3.11a}$$
$$y_{\rm cm} = -a\cos\theta. \tag{3.11b}$$

Show that

$$\dot{x}_{\rm cm} = \dot{x} + a\dot{\theta}\cos\theta, \tag{3.12a}$$
$$\dot{y}_{\rm cm} = a\dot{\theta}\sin\theta, \tag{3.12b}$$

and

$$\ddot{x}_{\rm cm} = \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta, \tag{3.13a}$$
$$\ddot{y}_{\rm cm} = a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta. \tag{3.13b}$$

Comparing Eqs. (3.9) and Eqs. (3.13) we learn that

$$\ddot{x}_{\rm cm} = 0, \tag{3.14a}$$
$$\sin\theta\Big[\ddot{y}_{\rm cm} + g\Big] = 0. \tag{3.14b}$$

Thus, $\ddot{y}_{\rm cm} = -g$ is the more general solution, and $\theta = 0$ is a trivial solution.

(f) Let us analyse the system for initial conditions: $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$, $\dot{x}(0) = 0$. Show that for this case $\dot{x}_{\rm cm}(0) = 0$ and

$$a(\cos\theta - \cos\theta_0) = \frac{1}{2}gt^2. \tag{3.15}$$

Plot θ as a function of time t. Interpret this solution.

(g) **To do:** The interpretation does not seem satisfactory. Is $m_1 = 0$ physical here?

15. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a_2. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a frictionless hoop of radius a_1. See Figure 15. For simplification, and at loss of generality, let us chose the motion of the pendulum in the plane containing the hoop.

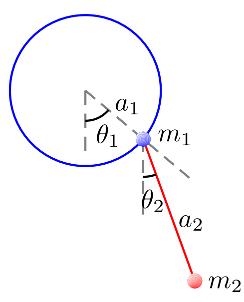

Figure 3.2: Problem 15.

(a) Determine the Lagrangian for the system to be

$$L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = \frac{1}{2}m_2 a_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 a_2^2\dot{\theta}_2^2 + m_2 a_1 a_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + m_2 g a_1\cos\theta_1 + m_2 g a_2\cos\theta_2. \tag{3.16}$$

(b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot\theta_1} = m_2 a_1^2 \dot\theta_1 + m_2 a_1 a_2 \dot\theta_2 \cos(\theta_1 - \theta_2), \tag{3.17a}$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 a_1 a_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_2 g a_1 \sin\theta_1, \tag{3.17b}$$

$$\frac{\partial L}{\partial \dot\theta_2} = m_2 a_2^2 \dot\theta_2 + m_2 a_1 a_2 \dot\theta_1 \cos(\theta_1 - \theta_2), \tag{3.17c}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 a_1 a_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_2 g a_2 \sin\theta_2. \tag{3.17d}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot\theta_1 + \omega_1^2 \sin\theta_1 + \frac{1}{\beta}\ddot\theta_2 \cos(\theta_1 - \theta_2) + \frac{1}{\beta}\dot\theta_2^2 \sin(\theta_1 - \theta_2) = 0, \tag{3.18a}$$

$$\ddot\theta_2 + \omega_2^2 \sin\theta_2 + \beta\ddot\theta_1 \cos(\theta_1 - \theta_2) - \beta\dot\theta_1^2 \sin(\theta_1 - \theta_2) = 0, \tag{3.18b}$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \tag{3.19}$$

Note that β is not an independent parameter. Also, observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

(d) In the small angle approximation show that the equations of motion reduce to

$$\ddot\theta_1 + \omega_1^2\theta_1 + \frac{1}{\beta}\ddot\theta_2 = 0, \tag{3.20a}$$

$$\ddot\theta_2 + \omega_2^2\theta_2 + \beta\ddot\theta_1 = 0. \tag{3.20b}$$

(e) Determine the solution for the initial conditions

$$\theta_1(0) = \theta_2(0) = \theta_{20}, \quad \dot\theta_1(0) = \dot\theta_2(0) = 0. \tag{3.21}$$

Interpret and expound your solution.

16. (**20 points.**) Consider the coplanar double pendulum in Figure 16.

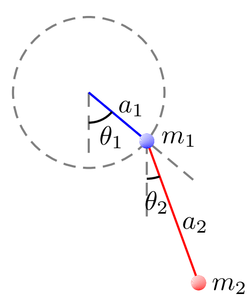

Figure 3.3: Problem 16.

(a) Write the Lagrangian for the system. in particular, show that the Lagrangian can be expressed in the form

$$L = L_1 + L_2 + L_{\text{int}}, \tag{3.22}$$

where

$$L_1 = \frac{1}{2}(m_1+m_2)a_1^2\dot{\theta}_1^2 + (m_1+m_2)ga_1\cos\theta_1, \tag{3.23a}$$

$$L_2 = \frac{1}{2}m_2a_2^2\dot{\theta}_2^2 + m_2ga_2\cos\theta_2, \tag{3.23b}$$

$$L_{\text{int}} = m_2a_1a_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2). \tag{3.23c}$$

(b) Determine the equations of motion for the system. Express them in the form

$$(m_1+m_2)a_1\ddot{\theta}_1 + (m_1+m_2)g\sin\theta_1 + m_2a_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2a_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) = 0, \tag{3.24a}$$

$$a_2\ddot{\theta}_2 + g\sin\theta_2 + a_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - a_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) = 0. \tag{3.24b}$$

(c) In the small angle approximation show that the equations of motion reduce to

$$\ddot{\theta}_1 + \omega_1^2\theta_1 + \frac{\alpha}{\beta}\ddot{\theta}_2 = 0, \tag{3.25a}$$

$$\ddot{\theta}_2 + \omega_2^2\theta_2 + \beta\ddot{\theta}_1 = 0, \tag{3.25b}$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \alpha = \frac{m_2}{m_1+m_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \tag{3.26}$$

Note that $0 \le \alpha \le 1$.

(d) Determine the solution for the initial conditions

$$\theta_1(0) = 0, \quad \theta_2(0) = 0, \quad \dot{\theta}_1(0) = 0, \quad \dot{\theta}_2(0) = \omega_0, \tag{3.27}$$

for $\alpha = 1/2$ and $\beta = 1$.

3.2 Lagrangian multiplier

1. (**20 points.**) Consider the function describing a paraboloid

$$f(x,y) = a(x^2+y^2). \tag{3.28}$$

A straight line on the xy plane, $y = mx + c$, can be interpreted as a condition of constraint

$$g(x,y) = y - mx - c = 0. \tag{3.29}$$

Let us determine the point on the line where the function $f(x,y)$ has an extremum value.

(a) Construct the function

$$F(x) = f(x, mx+c). \tag{3.30}$$

Using the extremum principle, $dF/dx = 0$, show that the point on the line where the function f is an extremum is

$$x = -\frac{mc}{1+m^2}, \quad y = \frac{c}{1+m^2}. \tag{3.31}$$

(b) Construct the function

$$h(x,y) = f(x,y) + \lambda g(x,y). \tag{3.32}$$

Evaluate $\boldsymbol{\nabla} h$, $\boldsymbol{\nabla} f$, and $\boldsymbol{\nabla} g$. Show that $\boldsymbol{\nabla} h = 0$ implies

$$x = \frac{\lambda m}{2a}, \quad y = -\frac{\lambda}{2a}. \tag{3.33}$$

Use this in the condition of constraint to derive

$$\lambda = -\frac{2ac}{1+m^2}. \tag{3.34}$$

Use the above expression for λ in Eq. (3.33) to find the point on the line where the function f is an extremum.

2. (**20 points.**) Spherical pendulum: Consider a pendulum that is suspended such that a mass m is able to move freely on the surface of a sphere of radius a (the length of the pendulum). The mass is then subject to the condition of constraint

$$F = \frac{1}{2}(x^2 + y^2 + z^2 - a^2) = 0, \tag{3.35}$$

where the factor of 1/2 is introduced anticipating cancellations. Consider the Lagrangian function

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} m \dot{\mathbf{r}}^2 - mgz - \lambda F. \tag{3.36}$$

(a) Evaluate the gradient $\boldsymbol{\nabla}$ of the condition of constraint. Show that

$$\boldsymbol{\nabla} F = \mathbf{r}. \tag{3.37}$$

(b) Using the Euler-Lagrange equations derive the equations of motion

$$m\ddot{\mathbf{r}} = -mg\hat{\mathbf{z}} - \lambda \mathbf{r}. \tag{3.38}$$

(c) Derive an expression for λ. In particular, show that it can be expressed in the form

$$-\lambda a = \hat{\mathbf{r}} \cdot \mathbf{N}. \tag{3.39}$$

Find $\mathbf{N}$. Give the physical interpretation of $\mathbf{N}$ using D'Alembert's principle.

(d) Show that the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p} = m\dot{\mathbf{r}}$ is the momentum of the particle, about the z-axis is conserved. That is,

$$\frac{d}{dt}(\hat{\mathbf{z}} \cdot \mathbf{L}) = 0. \tag{3.40}$$

Show that this also implies the conservation of the areal velocity

$$\frac{dS}{dt} = \frac{1}{2}(x\dot{y} - y\dot{x}), \tag{3.41}$$

where S is the area swept out.

(e) Show that

$$\frac{dF}{dt} = \mathbf{r} \cdot \dot{\mathbf{r}} = 0. \tag{3.42}$$

Using this derive the statement of conservation of energy,

$$\frac{dH}{dt} = 0, \qquad H = \frac{1}{2} m \dot{\mathbf{r}}^2 + mgz, \tag{3.43}$$

starting from the equation of motion in Eq. (3.38) and multiplying by $\dot{\mathbf{r}}$.

3. (**20 points.**) A (spherical) pendulum is suspended such that a mass m is able to move freely on the surface of a sphere of radius a (the length of the pendulum). The spherical pendulum is suitably described by the Lagrangian function

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2} m v^2 - mgz + \frac{1}{2}\left(\frac{r^2}{a^2} - 1\right) \mathbf{T} \cdot \mathbf{r}, \tag{3.44}$$

where $\mathbf{r}$ is the position vector with center of sphere as origin and $\mathbf{v} = d\mathbf{r}/dt$. Assume the acceleration due to gravity is downward, such that $\mathbf{g} = -g\,\hat{\mathbf{z}}$. Derive an expression for $\mathbf{T}$. In particular, give the physical interpretation of $\mathbf{T}$.

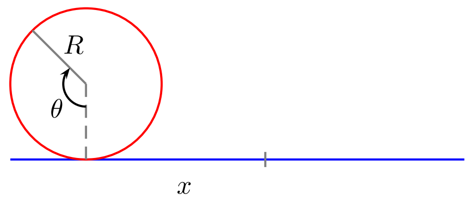

Figure 3.4: Problem 4.

4. (**20 points.**) Consider a wheel rolling on a horizontal surface. The following distinct types of motion are possible for the wheel:

$$\begin{aligned} x &< \theta R, && \text{slipping (e.g. in snow)}, \\ x &= \theta R, && \text{perfect rolling}, \\ x &> \theta R, && \text{sliding (e.g. on ice)}. \end{aligned} \tag{3.45}$$

Differentiation of the these relations leads to the characterizations, $v < \omega R$, $v = \omega R$, and $v > \omega R$, respectively, where $v = \dot{x}$ is the linear velocity and $\omega = \dot{\theta}$ is the angular velocity. Assuming the wheel is perfectly rolling, at a given instant of time, the tendency of motion could be to slip, to keep on perfectly rolling, or to slide.

Deduce that while perfectly rolling the relative motion of the point on the wheel that is in contact with the surface with respect to the surface is exactly zero. Thus, conclude that the force of friction on the wheel is zero. The analogy is a mass at rest on a horizontal surface. However, while perfectly rolling, it is possible to have the tendency to slip or slide without actually slipping of sliding. The analogy is that of a mass at rest under the action of an external force and the force of friction. In these cases the force of friction is that of static friction and it acts in the forward or backward direction.

In the following we differentiate between the following:

(a) Tendency of the wheel is to slip (without actually slipping) while perfectly rolling.

(b) Tendency of the wheel is to keep on perfectly rolling.

(c) Tendency of the wheel is to slide (without actually sliding) while perfectly rolling.

Deduce the direction of the force of friction in the above cases. Determine if the friction is working against linear acceleration or angular acceleration.

Perfect rolling involves the contraint $x = \theta R$. Thus, using the D'Alembert's principle and idea of Lagrange multiplier we can write the Lagrangian for a perfectly rolling wheel on a horizontal surface to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 - F_s (x - \theta R), \tag{3.46}$$

where m is the mass of the wheel, I is the moment of inertia of the wheel, and F_s is the Lagrangian multiplier. Using D'Alembert's principle give an interpretation for the Lagrangian multiplier F_s. What is the dimension of F_s? Infer the sign of F_s for the cases when the tendency of the wheel is to slip or slide while perfectly rolling.

5. (**20 points.**) Consider two discs of radii r_1 and r_2, and moment of inertia I_1 and I_2. Disc 1 is free to roll about an axis parallel to z axis passing through its center O_1. Similarly, disc 2 is free to roll about an axis parallel to z axis passing through its center O_2. Further, the center of disc 2 is free to move on a circle of radii $(r_1 + r_2)$. Let I_3 be the moment of inertia of disc 2 about the axis passing through O_1. See Figure 5. Assume gravity in the direction of z axis and no motion in the z direction so that gravity effects

are irrelevant. The two discs are in contact with sufficient friction between them such that the resultant motion leads to perfect rolling of the surfaces,

$$\theta_1 r_1 = \theta_2 r_2. \tag{3.47}$$

Here θ_1 and θ_2 are angular displacements of the respective discs about the axes O_1 and O_2. Further, the angular displacement of the axis O_2 about the axis O_1 is parametrized by the angular displacement α_2. Assume the discs are rolling under the action of no external torques.

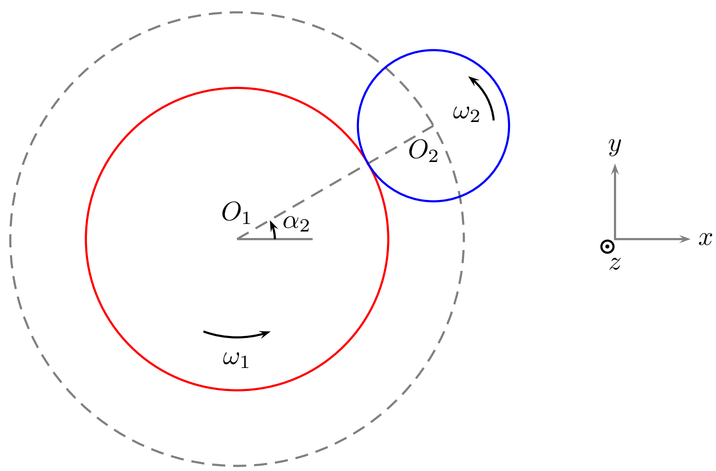

Figure 3.5: Problem 5.

(a) Show that the Lagrangian for this system in terms of the coordinates θ_1 and α_2, and their derivatives, is

$$\begin{aligned} L(\theta_1, \dot\theta_1, \alpha_2, \dot\alpha_2) &= \frac{1}{2} I_1 \dot\theta_1^2 + \frac{1}{2} I_2 \dot\theta_2^2 + \frac{1}{2} I_3 \dot\alpha_2^2 && (3.48\text{a}) \\ &= \frac{1}{2}\left(I_1 + I_2 \frac{r_1^2}{r_2^2}\right)\dot\theta_1^2 + \frac{1}{2} I_3 \dot\alpha_2^2, && (3.48\text{b}) \end{aligned}$$

where the equation of contraint has been used to replace θ_2. Determine the equations of motion to be

$$\left(I_1 + I_2 \frac{r_1^2}{r_2^2}\right)\ddot\theta_1 = 0 \tag{3.49}$$

and

$$I_3 \ddot\alpha_2 = 0. \tag{3.50}$$

These imply $\ddot\theta_1 = 0$ and $\ddot\alpha_2 = 0$ in the absence of external torque.

(b) Show that the Lagrangian for this system in terms of the coordinates θ_1, θ_2, and α_2, and their derivatives, is

$$L(\theta_1, \dot\theta_1, \theta_2, \dot\theta_2, \alpha_2, \dot\alpha_2) = \frac{1}{2} I_1 \dot\theta_1^2 + \frac{1}{2} I_2 \dot\theta_2^2 + \frac{1}{2} I_3 \dot\alpha^2 + \lambda(\theta_1 r_1 - \theta_2 r_2), \tag{3.51}$$

where the contraint has been introduced with Lagrange multiplier λ. Determine the equations of motion to be

$$\begin{aligned} I_1 \ddot\theta_1 &= \lambda r_1, && (3.52\text{a}) \\ I_2 \ddot\theta_2 &= -\lambda r_2, && (3.52\text{b}) \\ I_3 \ddot\alpha_2 &= 0. && (3.52\text{c}) \end{aligned}$$

Combine Eqs.(3.52a) and (3.52b) and show that it is consistent with Eq.(3.49).

(c) Which quantity relates to the Lagrange multiplier λ.

(d) In the absence of external torque and $\dot{\alpha}_2 = 0$ initially deduce that the center of mass of disc 2 is stationary.

Chapter 4

Stationary action principle

4.1 Review of Hamilton's principle

Problems

1. (**30 points.**) The motion of a particle of mass m near the Earth's surface is described by

$$\frac{d}{dt}(mv) = -mg, \tag{4.1}$$

where $v = dz/dt$ is the velocity in the upward z direction.

(a) Find the Lagrangian for this system that implies the equation of motion of Eq. (4.1) using Hamilton's principle of stationary action.

(b) Determine the canonical momentum for this system

(c) Determine the Hamilton $H(p, z)$ for this system.

(d) Determine the Hamilton equations of motion.

2. (**30 points.**) The motion of a particle of mass m undergoing simple harmonic motion is described by

$$\frac{d}{dt}(mv) = -kx, \tag{4.2}$$

where $v = dx/dt$ is the velocity in the x direction.

(a) Find the Lagrangian for this system that implies the equation of motion of Eq. (4.2) using Hamilton's principle of stationary action.

(b) Determine the canonical momentum for this system

(c) Determine the Hamiltonian $H(p, x)$ for this system.

(d) Determine the Hamilton equations of motion.

3. (**30 points.**) A non-relativistic charged particle of charge q and mass m in the presence of a known electric and magnetic field is described by

$$\frac{d}{dt}(m\mathbf{v}) = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \tag{4.3}$$

(a) Using

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{4.4a}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{4.4b}$$

find the Lagrangian for this system, that implies the equation of motion of Eq. (4.3), to be

$$L(\mathbf{x}, \mathbf{v}, t) = \frac{1}{2} m v^2 - q\phi + q\mathbf{v}\cdot\mathbf{A}, \tag{4.5}$$

using Hamilton's principle of stationary action.

(b) Determine the canonical momentum for this system

(c) Determine the Hamiltonian $H(\mathbf{x}, \mathbf{p}, t)$ for this system to be

$$H(\mathbf{x}, \mathbf{p}, t) = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2 + q\phi. \tag{4.6}$$

4. (**30 points.**) A relativistic charged particle of charge q and mass m in the presence of a known electric and magnetic field is described by

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{4.7}$$

(a) Find the Lagrangian for this system, that implies the equation of motion of Eq. (4.7), to be

$$L(\mathbf{x}, \mathbf{v}, t) = -mc^2\sqrt{1-\frac{v^2}{c^2}} - q\phi + q\mathbf{v}\cdot\mathbf{A}, \tag{4.8}$$

using Hamilton's principle of stationary action.

(b) Determine the canonical momentum for this system

(c) Determine the Hamilton $H(\mathbf{r}, \mathbf{p})$ for this system to be

$$H(\mathbf{x}, \mathbf{p}, t) = \sqrt{m^2c^4 + \left(\mathbf{p} - q\mathbf{A}\right)^2 c^2} + q\phi. \tag{4.9}$$

5. (**20 points.**) Verify, by substitution in Eqs. (4.4), that a plausible scalar and vector potential for constant (uniform in space and time) electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are

$$\phi = -\mathbf{r}\cdot\mathbf{E}, \tag{4.10a}$$

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}. \tag{4.10b}$$

Thus, show that

$$q\phi - q\mathbf{v}\cdot\mathbf{A} = -\mathbf{d}\cdot\mathbf{E} - \boldsymbol{\mu}\cdot\mathbf{B}, \tag{4.11}$$

where $\mathbf{d} = q\mathbf{r}$ is the electric dipole moment and $\boldsymbol{\mu} = \frac{q}{2}\mathbf{r}\times\mathbf{v} = \frac{q}{2m}\mathbf{L}$, with $\mathbf{L} = \mathbf{r}\times m\mathbf{v}$, is the magnetic dipole moment.

6. (**30 points.**) The Hamiltonian

$$H(\mathbf{x}, \mathbf{p}, t) = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2 + q\phi \tag{4.12}$$

describes a non-relativistic particle of charge q and mass m in the presence of a known electric and magnetic field. Find the Hamiltonian equations of motion to be

$$\frac{d\mathbf{x}}{dt} = \frac{1}{m}\left(\mathbf{p} - q\mathbf{A}\right), \tag{4.13a}$$

$$\frac{d\mathbf{p}}{dt} = -q\boldsymbol{\nabla}\phi + q(\boldsymbol{\nabla}\mathbf{A})\cdot\left(\mathbf{p} - q\mathbf{A}\right). \tag{4.13b}$$

Further, show that the above equations, in conjunction, imples the Lorentz force equation

$$\frac{d}{dt}\left(m\mathbf{v}\right) = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{4.14}$$

7. (**20 points.**) Consider the (time independent) Hamiltonian
$$H = H(x, p), \tag{4.15}$$
which satisfies the equations of motion
$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}. \tag{4.16}$$
Recollect that the Lagrangian, which will temporarily be called the x-Lagrangian here, is defined by the construction
$$L_x = p\frac{dx}{dt} - H \tag{4.17}$$
and implies the equations of motion
$$p = \frac{\partial L_x}{\partial \left(\frac{dx}{dt}\right)}, \qquad \frac{dp}{dt} = \frac{\partial L_x}{\partial x}. \tag{4.18}$$
Now, define the p-Lagrangian using the construction
$$L_p = -x\frac{dp}{dt} - H \tag{4.19}$$
and derive the equations of motion satisfied by the p-Lagrangian.

Comments: The opposite sign in the construction of the p-Lagrangian is motivated by the action principle, which does not care for a total derivative. You could use a specific Hamiltonian, for example that of a harmonic oscillator, as a guide.

8. (**20 points.**) Given a Lagrangian L, the Hamiltonian H is given by
$$H = \mathbf{p} \cdot \frac{d\mathbf{r}}{dt} - L, \tag{4.20}$$
where $\mathbf{p}$ is the canonical momentum. Evaluate
$$\frac{\partial H}{\partial \mathbf{v}}, \tag{4.21}$$
where $\mathbf{v}$ stands for $d\mathbf{r}/dt$.

9. (**20 points.**) The Hamiltonian is defined by the relation
$$H(p_i, q_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t). \tag{4.22}$$
Show that
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{4.23}$$
Under what circumstances is H interpreted as the energy of the system?

10. Consider the four-vector $x^\alpha = (ct, \mathbf{x})$. Here $\alpha = 0, 1, 2, 3$, such that $x^0 = ct$ and x^i are the three components of vector $\mathbf{x}$. The proper time s, that remains invariant under a Lorentz transformation, satisfies
$$-ds^2 = -c^2 dt^2 + d\mathbf{x} \cdot d\mathbf{x}. \tag{4.24}$$
Thus, derive the relation
$$\frac{1}{c}\frac{ds}{dt} = \sqrt{1 - \frac{v^2}{c^2}}, \tag{4.25}$$

where $\mathbf{v} = d\mathbf{x}/dt$. The energy E and momentum $\mathbf{p}$ of a particle of mass m is defined as

$$mc^2 \frac{dx^\alpha}{ds} = (E, \mathbf{p}c). \tag{4.26}$$

Find the explicit expressions for E and $\mathbf{p}$ in terms of $\mathbf{v}$, c, and m. Show that

$$\frac{dx^\alpha}{ds}\frac{dx_\alpha}{ds} = -1, \tag{4.27}$$

and use this to derive $E^2 = p^2c^2 + m^2c^4$.

11. (**30 points.**) Consider the Lagrangian

$$L = \frac{1}{2} m \left(\frac{d\mathbf{r}}{dt}\right)^2 - V(\mathbf{r}, t). \tag{4.28}$$

(a) Show that principle of stationary action with respect to $\delta\mathbf{r}$ implies Newton's second law

$$m\frac{d^2\mathbf{r}}{dt^2} = -\boldsymbol{\nabla} V. \tag{4.29}$$

(b) Show that principle of stationary action with respect to δt implies

$$\frac{d}{dt}\left[\frac{1}{2} m \left(\frac{d\mathbf{r}}{dt}\right)^2 + V\right] = \frac{\partial V}{\partial t}, \tag{4.30}$$

which for a static potential, $\partial V/\partial t = 0$, is the statement of conservation of energy.

(c) Show that the invariance of the total time derivative term, that gets contributions only from the end points, under an infinitesimal rigid rotation

$$\mathbf{r}' = \mathbf{r} - \delta\mathbf{r}, \qquad \delta\mathbf{r} = \delta\boldsymbol{\omega} \times \mathbf{r}, \tag{4.31}$$

implies the conservation of total angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

12. (**40 points.**) In terms of the Lagrangian function $L(\mathbf{r}, \mathbf{v}, t)$ the action functional $W[\mathbf{r}; t_1, t_2]$ is defined as

$$W[\mathbf{r}; t_1, t_2] = \int_{t_1}^{t_2} dt\, L(\mathbf{r}, \mathbf{v}, t), \tag{4.32}$$

where $\mathbf{v} = d\mathbf{r}/dt$.

(a) For arbitrary infinitesimal variations in the path

$$\bar{\mathbf{r}}(t) = \mathbf{r}(t) - \delta\mathbf{r}(t), \tag{4.33}$$

and infinitesimal general time transformation

$$\bar{t} = t - \delta t(t), \tag{4.34}$$

the change in action is given by

$$\begin{aligned} \delta W = & \int_{t_1}^{t_2} dt \frac{d}{dt}\Big[\mathbf{p}\cdot\delta\mathbf{r} - H\delta t\Big] \\ & + \int_{t_1}^{t_2} dt \left[\delta t\left(\frac{dH}{dt} + \frac{\partial L}{\partial t}\right) + \delta\mathbf{r}\cdot\left(\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}}\right)\right], \end{aligned} \tag{4.35}$$

where the canonical momentum and the Hamiltonian are defined as

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} \quad \text{and} \quad H = \mathbf{v}\cdot\mathbf{p} - L \tag{4.36}$$

respectively.

(b) The change in the action due to variations in path is captured in the functional derivative

$$\frac{\delta W}{\delta \mathbf{r}(t)} = \left(\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}}\right) + \Big[\delta(t-t_2) - \delta(t-t_1)\Big]\mathbf{p}. \tag{4.37}$$

The change in the action due to time transformation is captured in the functional derivative

$$\frac{\delta W}{\delta t(t)} = \left(\frac{dH}{dt} + \frac{\partial L}{\partial t}\right) - \Big[\delta(t-t_2) - \delta(t-t_1)\Big]H. \tag{4.38}$$

(c) In terms of the Hamiltonian the action takes the form

$$W[\mathbf{r},\mathbf{p};t_1,t_2] = \int_{t_1}^{t_2} dt \Big[\mathbf{v}\cdot\mathbf{p} - H(\mathbf{r},\mathbf{p},t)\Big]. \tag{4.39}$$

(d) Show that for for arbitrary infinitesimal variations in coordinate and momentum

$$\bar{\mathbf{r}}(t) = \mathbf{r}(t) - \delta\mathbf{r}(t) \quad \text{and} \quad \bar{\mathbf{p}}(t) = \mathbf{p}(t) - \delta\mathbf{p}(t), \tag{4.40}$$

and infinitesimal general time transformation, the change in action is given by

$$\begin{aligned} \delta W \;=\; & \int_{t_1}^{t_2} dt \frac{d}{dt}\Big[\mathbf{p}\cdot\delta\mathbf{r} - H\delta t\Big] \\ & + \int_{t_1}^{t_2} dt \left[\delta t\left(\frac{dH}{dt} - \frac{\partial H}{\partial t}\right) - \delta\mathbf{r}\cdot\left(\frac{d\mathbf{p}}{dt} + \frac{\partial H}{\partial \mathbf{r}}\right) + \delta\mathbf{p}\cdot\left(\frac{d\mathbf{r}}{dt} - \frac{\partial H}{\partial \mathbf{p}}\right)\right]. \end{aligned} \tag{4.41}$$

13. (**20 points.**) In terms of the Lagrangian function $L(\mathbf{r},\mathbf{v},t)$ the action $W[\mathbf{r};t_1,t_2]$ is defined as

$$W[\mathbf{r};t_1,t_2] = \int_{t_1}^{t_2} dt\, L(\mathbf{r},\mathbf{v},t), \tag{4.42}$$

where $\mathbf{v} = d\mathbf{r}/dt$. Find the change in the action under an infinitesimal general time transformation

$$\bar{t} = t - \delta t(t), \quad \delta t(t_1) = 0, \quad \delta t(t_1) = 0. \tag{4.43}$$

In paticular, evaluate the functional derivative

$$\frac{\delta W}{\delta t(t)} \tag{4.44}$$

for the variation $\delta t(t)$ satisfying the constraints of Eq. (4.43).

4.2 Symmetry and conservation principles

14. (**20 points.**) Consider infinitesimal rigid translation in space, described by

$$\delta\mathbf{r} = \delta\boldsymbol{\epsilon}, \quad \delta\mathbf{p} = 0, \quad \delta t = 0, \tag{4.45}$$

where $\delta\boldsymbol{\epsilon}$ is independent of position and time.

(a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta\boldsymbol{\epsilon}} = -\int_{t_1}^{t_2} dt \frac{\partial H}{\partial \mathbf{r}}. \tag{4.46}$$

(b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta \boldsymbol{\epsilon}} = \int_{t_1}^{t_2} dt \frac{d\mathbf{p}}{dt} = \mathbf{p}(t_2) - \mathbf{p}(t_1). \tag{4.47}$$

(c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$-\frac{\partial H}{\partial \mathbf{r}} = 0. \tag{4.48}$$

That is, when the Hamiltonian is independent of position. Or, when the force $\mathbf{F} = -\partial H/\partial \mathbf{r} = 0$.

(d) Deduce that the linear momentum is conserved, that is,

$$\mathbf{p}(t_1) = \mathbf{p}(t_2), \tag{4.49}$$

when the action has translation symmetry.

15. (**20 points.**) Consider infinitesimal rigid translation in time, described by

$$\delta \mathbf{r} = 0, \quad \delta \mathbf{p} = 0, \quad \delta t = \delta \epsilon, \tag{4.50}$$

where $\delta\epsilon$ is independent of position and time.

(a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta \epsilon} = -\int_{t_1}^{t_2} dt \frac{\partial H}{\partial t}. \tag{4.51}$$

(b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta \epsilon} = -\int_{t_1}^{t_2} dt \frac{dH}{dt} = -H(t_2) + H(t_1). \tag{4.52}$$

(c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$-\frac{\partial H}{\partial t} = 0. \tag{4.53}$$

That is, when the Hamiltonian is independent of time.

(d) Deduce that the Hamiltonian is conserved, that is,

$$H(t_1) = H(t_2), \tag{4.54}$$

when the action has translation symmetry.

16. (**20 points.**) A general rotation in 3-dimensions can be written in terms of consecutive rotations about x, y, and z axes,

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{4.55}$$

For infinitesimal rotations we use

$$\cos\theta_i \sim 1, \tag{4.56a}$$

$$\sin\theta_i \sim \theta_i \to \delta\theta_i, \tag{4.56b}$$

to obtain

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & \delta\theta_3 & -\delta\theta_2 \\ -\delta\theta_3 & 1 & \delta\theta_1 \\ \delta\theta_2 & -\delta\theta_1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{4.57}$$

Show that this corresponds to the vector relation

$$\mathbf{r}' = \mathbf{r} - \delta\boldsymbol{\theta} \times \mathbf{r}, \tag{4.58}$$

where

$$\mathbf{r} = x_1\hat{\mathbf{x}} + x_2\hat{\mathbf{y}} + x_3\hat{\mathbf{z}}, \tag{4.59a}$$

$$\delta\boldsymbol{\theta} = \delta\theta_1\hat{\mathbf{x}} + \delta\theta_2\hat{\mathbf{y}} + \delta\theta_3\hat{\mathbf{z}}. \tag{4.59b}$$

As a particular example, verify that a rotation about the direction $\hat{\mathbf{z}}$ by an infinitesimal (azimuth) angle $\delta\phi$ is described by

$$\delta\boldsymbol{\theta} = \hat{\mathbf{z}}\,\delta\phi. \tag{4.60}$$

The corresponding infinitesimal transformation in $\mathbf{r}$ is given by

$$\delta\mathbf{r} = \delta\phi\,\hat{\mathbf{z}} \times \mathbf{r} = \hat{\boldsymbol{\phi}}\,\rho\delta\phi, \tag{4.61}$$

where ρ and ϕ are the cylindrical coordinates defined as

$$\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \quad \text{and} \quad |\hat{\mathbf{z}} \times \mathbf{r}| = \rho. \tag{4.62}$$

Observe that, in rectangular coordinates $\rho\hat{\boldsymbol{\phi}} = x\hat{\mathbf{y}} - y\hat{\mathbf{x}}$.

17. (**20 points.**) Consider infinitesimal rigid rotation, described by

$$\delta\mathbf{r} = \delta\boldsymbol{\omega} \times \mathbf{r}, \quad \delta\mathbf{p} = \delta\boldsymbol{\omega} \times \mathbf{p}, \quad \delta t = 0, \tag{4.63}$$

where $d\delta\boldsymbol{\omega}/dt = 0$.

(a) Show that the variation in the action under the above rotation is

$$\frac{\delta W}{\delta\boldsymbol{\omega}} = \int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial L}{\partial\mathbf{r}} + \mathbf{p} \times \frac{\partial L}{\partial\mathbf{p}}\right] \tag{4.64}$$

or

$$\frac{\delta W}{\delta\boldsymbol{\omega}} = -\int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial H}{\partial\mathbf{r}} + \mathbf{p} \times \frac{\partial H}{\partial\mathbf{p}}\right]. \tag{4.65}$$

(b) Show, separately, that the change in the action under the above rotation is also given by

$$\frac{\delta W}{\delta\boldsymbol{\omega}} = \int_{t_1}^{t_2} dt \frac{d\mathbf{L}}{dt} = \mathbf{L}(t_2) - \mathbf{L}(t_1), \tag{4.66}$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum.

(c) The system is defined to have rotational symmetry when the action does not change under rigid rotation. Show that a system has rotation symmetry when

$$\mathbf{r} \times \frac{\partial L}{\partial\mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial L}{\partial\mathbf{p}} = 0, \tag{4.67}$$

or

$$\mathbf{r} \times \frac{\partial H}{\partial\mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial H}{\partial\mathbf{p}} = 0. \tag{4.68}$$

Show that this corresponds to

$$\frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \phi} = 0, \tag{4.69}$$

or

$$\frac{\partial H}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial H}{\partial \phi} = 0. \tag{4.70}$$

That is, when the Lagrangian is independent of angular coordinates θ and ϕ.

(d) Deduce that the anglular momentum is conserved, that is,

$$\mathbf{L}(t_1) = \mathbf{L}(t_2), \tag{4.71}$$

when the action has rotational symmetry.

18. (**20 points.**) Noether's theorem, in the context of rotational symmetry, states that if the Lagrangian does not change under an infinitesimal rigid rotation, then the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved. Prove that the converse of Noether's theorem is also true. For simplicity consider velocity independent potentials.

Chapter 5

Canonical transformation

5.1 Hamilton-Jacobi equation

1. (**40 points.**) The Hamiltonian for the motion of a particle of mass m in a constant gravitational field $\mathbf{g} = -g\hat{\mathbf{z}}$ is

$$H(z,p,t) = \frac{p^2}{2m} + mgz. \tag{5.1}$$

(a) Show that the Hamilton equations of motion are

$$\frac{dz}{dt} = \frac{p}{m}, \tag{5.2a}$$

$$\frac{dp}{dt} = -mg. \tag{5.2b}$$

(b) Show that the Hamilton-Jacobi equation

$$-\frac{\partial W}{\partial t} = H\left(z, \frac{\partial W}{\partial z}, t\right), \tag{5.3}$$

in terms of Hamilton's principal function $W(z,t)$ is given by

$$-\frac{\partial W}{\partial t} = \frac{1}{2m}\left(\frac{\partial W}{\partial z}\right)^2 + mgz. \tag{5.4}$$

Further, show that

$$W(z,t) = -Et - \frac{2}{3}\frac{\sqrt{2m}}{mg}(E - mgz)^{\frac{3}{2}} \tag{5.5}$$

is a solution to the Hamilton-Jacobi equation up to a constant.

(c) Hamilton's principal function allows us to identify canonical transformations $Q = Q(z,p,t)$ and $P = P(z,p,t)$, such that

$$\frac{\partial W}{\partial z} = p, \qquad \frac{\partial W}{\partial Q} = -P, \qquad \frac{\partial W}{\partial t} = -H, \tag{5.6a}$$

$$\frac{\partial W}{\partial p} = 0, \qquad \frac{\partial W}{\partial P} = 0, \tag{5.6b}$$

with the feature that the new coordinates are constants of motion,

$$\frac{dQ}{dt} = 0 \qquad \text{and} \qquad \frac{dP}{dt} = 0. \tag{5.7}$$

To this end, choose $Q = E$ and then evaluate

$$P = -\frac{\partial W}{\partial Q} = t + \frac{p}{mg}. \tag{5.8}$$

Hint: Use $p = \frac{\partial W}{\partial z}$.

(d) Show that

$$\begin{aligned} Q &= \frac{p^2}{2m} + mgz, && (5.9a) \\ P &= t + \frac{p}{mg}, && (5.9b) \end{aligned}$$

is a canonical transformation. That is, show that $[Q,P]^{\text{P.B.}}_{z,p} = 1$. Further, verify that

$$\begin{aligned} \frac{dQ}{dt} &= 0, && (5.10a) \\ \frac{dP}{dt} &= 0, && (5.10b) \\ K(Q,P,t) &= H(z,p,t) + \frac{\partial W}{\partial t} = 0. && (5.10c) \end{aligned}$$

5.2 Poisson braket

1. (**40 points.**) Type notes dated 2022 Mar 29.

5.2.1 Lie Algebra of Poisson braket

1. (**40 points.**) For two functions

$$\begin{aligned} A &= A(\mathbf{x},\mathbf{p},t), && (5.11a) \\ B &= B(\mathbf{x},\mathbf{p},t), && (5.11b) \end{aligned}$$

the Poisson braket with respect to the canonical variables $\mathbf{x}$ and $\mathbf{p}$ is defined as

$$\big[A,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \equiv \frac{\partial A}{\partial \mathbf{x}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{x}}. \tag{5.12}$$

Show that the Poisson braket satisfies the conditions for a Lie algebra. That is, show that

(a) Antisymmetry:

$$\big[A,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = -\big[B,A\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{5.13}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[aA + bB, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = a\big[A,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + b\big[B,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{5.14}$$

Further show that

$$\big[AB, C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = A\big[B,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} B. \tag{5.15}$$

(c) Jacobi's identity:

$$\Big[A, \big[B,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \Big[B, \big[C,A\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \Big[C, \big[A,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{5.16}$$

2. (**40 points.**) Show that the commutator of two matrices,
$$\big[\mathbf{A},\mathbf{B}\big] \equiv \mathbf{AB} - \mathbf{BA}, \tag{5.17}$$
satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that
   (a) Antisymmetry:
$$\big[\mathbf{A},\mathbf{B}\big] = -\big[\mathbf{B},\mathbf{A}\big]. \tag{5.18}$$
   (b) Bilinearity: (a and b are numbers.)
$$\big[a\mathbf{A}+b\mathbf{B},\mathbf{C}\big] = a\big[\mathbf{A},\mathbf{C}\big] + b\big[\mathbf{B},\mathbf{C}\big]. \tag{5.19}$$
   Further show that
$$\big[\mathbf{AB},\mathbf{C}\big] = \mathbf{A}\big[\mathbf{B},\mathbf{C}\big] + \big[\mathbf{A},\mathbf{C}\big]\mathbf{B}. \tag{5.20}$$
   (c) Jacobi's identity:
$$\big[\mathbf{A},\big[\mathbf{B},\mathbf{C}\big]\big] + \big[\mathbf{B},\big[\mathbf{C},\mathbf{A}\big]\big] + \big[\mathbf{C},\big[\mathbf{A},\mathbf{B}\big]\big] = 0. \tag{5.21}$$

3. (**40 points.**) Show that the vector product of two vectors, in this problem denoted using
$$\big[\mathbf{A},\mathbf{B}\big]_v \equiv \mathbf{A}\times\mathbf{B}, \tag{5.22}$$
satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that
   (a) Antisymmetry:
$$\big[\mathbf{A},\mathbf{B}\big]_v = -\big[\mathbf{B},\mathbf{A}\big]_v. \tag{5.23}$$
   (b) Bilinearity: (a and b are numbers.)
$$\big[a\mathbf{A}+b\mathbf{B},\mathbf{C}\big]_v = a\big[\mathbf{A},\mathbf{C}\big]_v + b\big[\mathbf{B},\mathbf{C}\big]_v. \tag{5.24}$$
   Further show that
$$\big[\mathbf{A}\times\mathbf{B},\mathbf{C}\big]_v = \mathbf{A}\times\big[\mathbf{B},\mathbf{C}\big]_v + \big[\mathbf{A},\mathbf{C}\big]_v\times\mathbf{B}. \tag{5.25}$$
   (c) Jacobi's identity:
$$\big[\mathbf{A},\big[\mathbf{B},\mathbf{C}\big]_v\big]_v + \big[\mathbf{B},\big[\mathbf{C},\mathbf{A}\big]_v\big]_v + \big[\mathbf{C},\big[\mathbf{A},\mathbf{B}\big]_v\big]_v = 0. \tag{5.26}$$

4. (**40 points.**) Construct a problem on Heisenberg group, Weyl algebra, Bergman-Segal space.

5. (**40 points.**) (Refer Sec. 21 Dirac's QM book.)
   The product rule for Poisson braket can be stated in the following different forms:
$$\big[A_1A_2,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = A_1\big[A_2,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A_1,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2, \tag{5.27a}$$
$$\big[A,B_1B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = B_1\big[A,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2. \tag{5.27b}$$
   (a) Thus, evaluate, in two different ways,
$$\begin{aligned}\big[A_1A_2,B_1B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} &= A_1B_1\big[A_2,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + A_1\big[A_2,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2\\ &\quad + B_1\big[A_1,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2 + \big[A_1,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2A_2,\end{aligned} \tag{5.28a}$$
$$\begin{aligned}\big[A_1A_2,B_1B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} &= B_1A_1\big[A_2,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + B_1\big[A_1,B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2\\ &\quad + A_1\big[A_2,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2 + \big[A_1,B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2B_2.\end{aligned} \tag{5.28b}$$

(b) Subtracting these results, obtain

$$(A_1B_1 - B_1A_1)\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}(A_2B_2 - B_2A_2). \tag{5.29}$$

Thus, using the definition of the commutation relation,

$$\big[A, B\big] \equiv AB - BA, \tag{5.30}$$

obtain the relation

$$\big[A_1, B_1\big]\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\big[A_2, B_2\big]. \tag{5.31}$$

(c) Since this condition holds for A_1 and B_1 independent of A_2 and B_2, conclude that

$$\begin{aligned}\big[A_1, B_1\big] &= i\hbar\big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, && (5.32a)\\ \big[A_2, B_2\big] &= i\hbar\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, && (5.32b)\end{aligned}$$

where $i\hbar$ is necessarily a constant, independent of A_1, A_2, B_1, and B_2. This is the connection between the commutator braket in quantum mechanics and the Poisson braket in classical mechanics. If A's and B's are numbers, then, because their commutation relation is equal to zero, we necessairily have $\hbar = 0$. But, if the commutation relation of A's and B's is not zero, then finite values of $\hbar$ is allowed.

(d) Here the imaginary number $i = \sqrt{-1}$. Show that the constant $\hbar$ is a real number if we presume the Poisson braket to be real, and require the construction

$$C = \frac{1}{i}(AB - BA) \tag{5.33}$$

to be Hermitian. Experiment dictates that $\hbar = h/2\pi$, where

$$h \sim 6.63 \times 10^{-34}\,\text{J·s} \tag{5.34}$$

is the Planck's constant with dimensions of action.

6. (**20 points.**) Given F and G are constants of motion, that is

$$\big[F, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0 \qquad \text{and} \qquad \big[G, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{5.35}$$

Then, using Jacobi's identity, show that $\big[F, G\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}$ is also a constant of motion. Thus, conclude the following:

(a) If L_x and L_y are constants of motion, then L_z is also a constant of motion.

(b) If p_x and L_z are constants of motion, then p_y is also a constant of motion.

7. (Refer Goldstein, Sec. 9.5.) Hamiltonian for the motion of a ball (along the radial direction) near the surface of Earth is given by

$$H(z, p_z) = \frac{p_z^2}{2m} - mgz. \tag{5.36}$$

(a) Determine the equations of motions using

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_z} \qquad \text{and} \qquad \frac{dp_z}{dt} = -\frac{\partial H}{\partial z}. \tag{5.37}$$

Then, solve the coupled differential equations to find the familiar elementary solution

$$z = z_0 + \frac{p_0}{m}t + \frac{1}{2}gt^2. \tag{5.38}$$

(b) Next, determine the equations of motion using

$$\big[z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{\partial H}{\partial p_z} \qquad \text{and} \qquad \big[p_z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = -\frac{\partial H}{\partial z}. \tag{5.39}$$

Then, using

$$z = z_0 + t\big[z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0} + \frac{1}{2}t^2\big[\big[z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0} + \cdots \tag{5.40}$$

rederive the elementary solution. Here 0 in the subscripts refers to the initial conditions at $t = 0$.

8. Harmonic oscillations are described by the Hamiltonian

$$H(x, p) = \frac{1}{2}p^2 + \frac{1}{2}x^2. \tag{5.41}$$

(a) Determine the equations of motions using

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \qquad \text{and} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}. \tag{5.42}$$

Then, solve the coupled differential equations to find the solution

$$x = x_0 \cos t + p_0 \sin t, \tag{5.43}$$

where x_0 and p_0 are given using the intial conditions at $t = 0$.

(b) Next, determine the equations of motion using

$$\big[x, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{\partial H}{\partial p} \qquad \text{and} \qquad \big[p, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = -\frac{\partial H}{\partial x}. \tag{5.44}$$

Show that

$$\big[\ldots\big[\big[x, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \ldots\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \begin{cases} \frac{1}{i}i^N p, & \text{for number of commutators, } N\text{, being odd,} \\ i^N x, & \text{for number of commutators, } N\text{, being even.} \end{cases} \tag{5.45}$$

Then, using

$$x = x_0 + t\big[x, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0} + \frac{1}{2}t^2\big[\big[x, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0} + \cdots \tag{5.46}$$

rederive the solution. Here 0 in the subscripts refers to the initial conditions at $t = 0$.

5.3 Charge in a magnetic field

1. (**30 points.**) Hamiltonian for a charge particle of mass m and charge q in a magnetic field $\mathbf{B}$ is given by

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2, \tag{5.47}$$

where

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{5.48}$$

Let

$$\frac{\partial \mathbf{A}}{\partial t} = 0. \tag{5.49}$$

Further, the magnetic vector potential $\mathbf{A}(\mathbf{x}, t)$ is presumed to be independent of $\mathbf{p}$.

(a) Show that the Hamilton equations of motion leads to the equations, using ($\mathbf{v} = d\mathbf{x}/dt$)

$$m\mathbf{v} = \mathbf{p} - q\mathbf{A}, \tag{5.50a}$$

$$\frac{d\mathbf{p}}{dt} = q(\boldsymbol{\nabla}\mathbf{A}) \cdot \mathbf{v}. \tag{5.50b}$$

Show that the above equations in conjunction imply the familiar equation

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}. \tag{5.51}$$

(b) Evaluate the Poisson braket

$$\big[\mathbf{x}, \mathbf{x}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{5.52}$$

(c) Evaluate the Poisson braket

$$\big[\mathbf{x}^i, \mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{1}{m}\mathbf{1}^{ij}. \tag{5.53}$$

(d) Evaluate the Poisson braket

$$\big[\mathbf{x}^i, \mathbf{p}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \mathbf{1}^{ij}. \tag{5.54}$$

(e) Evaluate the Poisson braket

$$\big[m\mathbf{v}^i, m\mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = q(\boldsymbol{\nabla}^i\mathbf{A}^j - \boldsymbol{\nabla}^j\mathbf{A}^i). \tag{5.55}$$

Verify that

$$(\boldsymbol{\nabla}^i\mathbf{A}^j - \boldsymbol{\nabla}^j\mathbf{A}^i) = \varepsilon^{ijk}\mathbf{B}^k = -\mathbf{1} \times \mathbf{B}. \tag{5.56}$$

Poisson bracket in classical mechanics has direct correspondence to commutation relation in quantum mechanics through the factor $i\hbar$, which conforms with experiments and balances the dimensions. Then, we can write

$$\big[m\mathbf{v}^i, m\mathbf{v}^j\big] = i\hbar q\varepsilon^{ijk}\mathbf{B}^k \tag{5.57}$$

or

$$m\mathbf{v} \times m\mathbf{v} = i\hbar q\mathbf{B}, \tag{5.58}$$

using the fact that the commutator and the vector product satisfies the same Lie algebra as that of Poisson bracket.

(f) Evaluate the Poisson braket

$$\big[\mathbf{p}^i, \mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{q}{m}\boldsymbol{\nabla}^i\mathbf{A}^j. \tag{5.59}$$

Using the antisymmetry property of the Poisson bracket conclude that

$$\big[\mathbf{p}^i, \mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[\mathbf{v}^i, \mathbf{p}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{q}{m}(\boldsymbol{\nabla}^i\mathbf{A}^j - \boldsymbol{\nabla}^j\mathbf{A}^i). \tag{5.60}$$

Thus, show that

$$\big[\mathbf{p}^i, \mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[\mathbf{v}^i, \mathbf{p}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = -\frac{q}{m}\mathbf{1} \times \mathbf{B} \equiv \frac{q}{m}\varepsilon^{ijm}\mathbf{B}^m. \tag{5.61}$$

Deduce the corresponding expression in quantum mechanics to be

$$\mathbf{p} \times \mathbf{v} + \mathbf{v} \times \mathbf{p} = i\hbar\frac{q}{m}\mathbf{B}. \tag{5.62}$$

(g) Evaluate the Poisson braket

$$\big[\mathbf{p}, \mathbf{p}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{5.63}$$

5.4 Infinitesimal canonical transformation

1. The generator for rotations satisfies the equations

$$\boldsymbol{\nabla}_{\mathbf{r}} G = -\delta\boldsymbol{\omega} \times \mathbf{p}, \tag{5.64a}$$
$$\boldsymbol{\nabla}_{\mathbf{p}} G = \delta\boldsymbol{\omega} \times \mathbf{r}. \tag{5.64b}$$

Show that

$$G = \delta\boldsymbol{\omega} \cdot \mathbf{L} \tag{5.65}$$

is a solution for the generator, where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum,

Chapter 6

Kepler problem

6.1 Ellipse

Refer Notes on Quantum Mechanics.

6.2 Conserved quantities

1. (**20 points.**) Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the postions of masses m_1 and m_2, respectively, with repect to an inertial frame. The gravitational interaction energy between the two masses is given by

$$\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{6.1}$$

Assume that the masses have no other internal or external interaction. The position of the center of mass $\mathbf{R}$ is defined by

$$(m_1 + m_2)\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 \tag{6.2}$$

and the relative position between the masses is given by

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1. \tag{6.3}$$

What is the motion of the center of mass $\mathbf{R}$ with respect to the position $\mathbf{r}_1$.

(a) Stays fixed.

(b) Circular.

(c) Elliptic (or conic section).

(d) None of the above.

Hint: The positions represented by the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{R}$ are collinear. Further, $\mathbf{r}$ describes an ellipse.
Solution: Show that

$$\mathbf{R} - \mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}, \tag{6.4a}$$

$$\mathbf{R} - \mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{6.4b}$$

Then, using $\mathbf{r}$ is elliptic, conclude that $\mathbf{R} - \mathbf{r}_i$ describes an ellipse whose length is scaled down by the factor $m_i/(m_1 + m_2)$.

2. (Refer Schwinger, chapter 9) The Hamiltonian for a Kepler problem is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{\alpha}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{6.5}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the two constituent particles of masses m_1 and m_2.

(a) Introduce the coordinates representing the center of mass, relative position, total momentum, and relative momentum:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2}, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2}, \tag{6.6}$$

respectively, to rewrite the Hamiltonian as

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} - \frac{\alpha}{r}, \tag{6.7}$$

where

$$M = m_1 + m_2, \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{6.8}$$

(b) Show that Hamilton's equations of motion are given by

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{P}}{M}, \quad \frac{d\mathbf{P}}{dt} = 0, \quad \frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\mu}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\alpha\mathbf{r}}{r^3}. \tag{6.9}$$

(c) Verify that the Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times \mathbf{L}}{\mu\alpha}, \tag{6.10}$$

are the three constants of motion for the Kepler problem. That is, show that

$$\frac{dH}{dt} = 0, \quad \frac{d\mathbf{L}}{dt} = 0, \quad \frac{d\mathbf{A}}{dt} = 0. \tag{6.11}$$

3. (**20 points.**) The Hamiltonian for a Kepler problem (or a classical hydrogenic atom) is

$$H(\mathbf{r}, \mathbf{p}) = \frac{p^2}{2m} - \frac{\alpha}{r}, \tag{6.12}$$

where $r = |\mathbf{r}|$ and $p = |\mathbf{p}|$. The Hamilton's equations of motion for the Kepler are

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{m}, \qquad \frac{d\mathbf{p}}{dt} = -\alpha\frac{\mathbf{r}}{r^3}. \tag{6.13}$$

The Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{m\alpha}\mathbf{p} \times \mathbf{L}, \tag{6.14}$$

are the three constants of motion for a Kepler problem. Under the special circumstance when $r = |\mathbf{r}|$ is also a conserved quantity, that is,

$$\frac{dr}{dt} = 0, \tag{6.15}$$

we have the case of circular motion. Evaluate the Laplace-Runge-Lenz vector for this case of circular orbit.

4. (**50 points.**) The Hamiltonian for a Kepler problem is

$$H(\mathbf{r},\mathbf{p}) = \frac{p^2}{2\mu} - \frac{\alpha}{r}. \tag{6.16}$$

The Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, and the axial vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p}\times\mathbf{L}}{\mu\alpha}, \tag{6.17}$$

are conserved quantities for a Kepler problem.

(a) Show that

$$\mathbf{W} = \frac{\mu\alpha}{L^2}\mathbf{A}\times\mathbf{L} \tag{6.18}$$

is also a conserved quantity. That is, show that $d\mathbf{W}/dt = 0$. Thus, together, the vectors $\mathbf{L}$, $\mathbf{A}$, and $\mathbf{W}$, form an orthogonal set that remain fixed in time. Show that the vector $\mathbf{W}$ can be expressed in the form

$$\mathbf{W} = \mathbf{p} + \frac{\mu\alpha}{L^2}\hat{\mathbf{r}}\times\mathbf{L}. \tag{6.19}$$

Further, show that

$$W = \mu\alpha\frac{A}{L}. \tag{6.20}$$

(b) Determine the components of the momentum $\mathbf{p}$ along these orthogonal vectors by evaluating $(\mathbf{p}\cdot\hat{\mathbf{L}})$, $(\mathbf{p}\cdot\hat{\mathbf{A}})$, and $(\mathbf{p}\cdot\hat{\mathbf{W}})$. Thus, construct the momentum $\mathbf{p}$ in the form

$$\mathbf{p} = (\mathbf{p}\cdot\hat{\mathbf{L}})\,\hat{\mathbf{L}} + (\mathbf{p}\cdot\hat{\mathbf{A}})\,\hat{\mathbf{A}} + (\mathbf{p}\cdot\hat{\mathbf{W}})\,\hat{\mathbf{W}}. \tag{6.21}$$

Hint: Show that

$$\mathbf{p}\cdot\mathbf{L} = 0, \qquad \mathbf{p}\cdot\mathbf{A} = \mathbf{p}\cdot\hat{\mathbf{r}}, \qquad \mathbf{p}\cdot\mathbf{W} = \frac{p^2}{2} + \mu H. \tag{6.22}$$

(c) It is well known that the position $\mathbf{r}$ traverses an ellipse about the origin. This is the content of Kepler's first law of motion. Show that the momentum $\mathbf{p}$ traverses a circle about a fixed point $\mathbf{p}_0$. That is, show that the momentum $\mathbf{p}$ satisfies the equation of a circle,

$$|\mathbf{p} - \mathbf{p}_0| = q. \tag{6.23}$$

Hint: Rewrite the expression for $(\mathbf{p}\cdot\hat{\mathbf{W}})$ in the form $\mathbf{p}\cdot\mathbf{p} - 2\mathbf{p}\cdot\mathbf{W} + \mathbf{W}\cdot\mathbf{W} = W^2 - 2\mu H$.

(d) Determine the vector $\mathbf{p}_0$ representing the center of this circle, and find the radius q of this circle. Verify that the center $\mathbf{p}_0$ is a conserved quantity.
Solution: $\mathbf{p}_0 = \mathbf{W}$ and $q = \mu\alpha/L$.

(e) Show that when the position $\mathbf{r}$ traverses a circle ($A = 0$) the center of the circle traversed by momentum $\mathbf{p}$ is the origin.

6.3 Kepler orbits

1. (**20 points.**) Starting from the Lagrangian for the Kepler problem,

$$L(\mathbf{r},\mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}, \tag{6.24}$$

derive Kepler's first law of planetary motion, which states that the orbit of a planet is a conic section. In particular, derive

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)}, \tag{6.25}$$

which is the equation of a conic section in terms of the eccentricity e and a distance r_0. The distance r_0 is characterized by the fact that the effective potential

$$U_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r} \tag{6.26}$$

is minimum at r_0. We used the definitions, $L_z = \mu r^2 \dot{\phi}$,

$$r_0 = \frac{L_z^2}{\mu\alpha}, \qquad U_{\text{eff}}(r_0) = -\frac{\alpha}{2r_0}, \qquad e = \sqrt{1 - \frac{E}{U_{\text{eff}}(r_0)}}. \tag{6.27}$$

Thus, the orbit of a planet is completely determined by the energy E and the angular momentum L_z, which are constants of motion.

2. (**20 points.**) In the Kepler problem the orbit of a planet is a conic section

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)} \tag{6.28}$$

expressed in terms of the eccentricity e and distance r_0. Determine the constant ϕ_0 to be 0 by requiring the initial condition

$$r(0) = \frac{r_0}{1+e}. \tag{6.29}$$

This leads to

$$r(\pi) = \frac{r_0}{1-e}. \tag{6.30}$$

The distance r_0 is characterized by the fact that the effective potential

$$U_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r} \tag{6.31}$$

is minimum at r_0. We used the definitions

$$r_0 = \frac{L_z^2}{\mu\alpha}, \qquad U_{\text{eff}}(r_0) = -\frac{\alpha}{2r_0}, \qquad e = \sqrt{1 - \frac{E}{U_{\text{eff}}(r_0)}}. \tag{6.32}$$

Thus, the orbit of a planet is completely determined by the energy E and the angular momentum L_z, which are constants of motion. The statement of conservation of angular momentum can be expressed in the form

$$dt = \frac{\mu}{L_z} r^2 d\phi, \tag{6.33}$$

which is convenient for evaluating the time elapsed in the motion. For the case of elliptic orbit, $U_{\text{eff}}(r_0) < E < 0$, show that the time period is given by

$$T = \frac{\mu}{L_z}\int_0^{2\pi} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{\mu r_0^2}{L_z}\frac{2\pi}{(1-e^2)^{\frac{3}{2}}}. \tag{6.34}$$

Show that at point '2' in Figure 2

$$\phi = \frac{\pi}{2}, \quad \text{and} \quad r = r_0. \tag{6.35}$$

The time taken to go from '1' to '2' is given by (need not be proved here)

$$t_{1\to 2} = \frac{\mu}{L_z}\int_0^{\frac{\pi}{2}} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{T}{4}\left(\frac{4}{\pi}\tan^{-1}\sqrt{\frac{1-e}{1+e}} - \frac{2e}{\pi}\sqrt{1-e^2}\right). \tag{6.36}$$

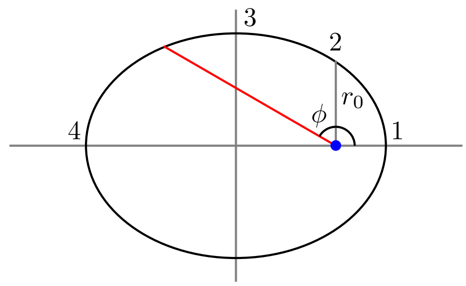

Figure 6.1: Elliptic orbit

Evaluate $t_{1\to 2}$ for $e = 0$ and $e = 1$. Show that at point '3' in Figure 2

$$\phi = \pi - \tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right), \quad \text{and} \quad r = a. \tag{6.37}$$

The time taken to go from '1' to '3' is given by (need not be proved here)

$$t_{1\to 3} = \frac{\mu}{L_z}\int_0^{\pi-\tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right)} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{T}{4}\left(1 - \frac{2e}{\pi}\right). \tag{6.38}$$

Similarly, the time taken to go from '3' to '4' is given by (need not be proved here)

$$t_{3\to 4} = \frac{\mu}{L_z}\int_{\pi-\tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right)}^{\pi} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{T}{4}\left(1 + \frac{2e}{\pi}\right). \tag{6.39}$$

Evaluate the time elapsed in the above cases for $e \to 0$ and $e \to 1$. The eccentricity e of Earth's orbit is 0.0167 and timeperiod T is 365 days. Thus, calculate

$$t_{1\to 3} - t_{1\to 2} \tag{6.40}$$

for Earth in units of days.
Solution: ~ 1 day for Earth.

3. (**20 points.**) Refer to the essay by J. M. Luttinger titled 'On "negative" mass in the theory of gravitation' in 1951.

   (a) Reproduce all the equations in the essay.

   (b) Critically assess the logic of the arguments in the essay.

6.4 Precession of the perihelion

1. (**20 points.**) Resource: Lecture dated 2021 April 13, available at

   https://youtu.be/VGfxAMMkvM4.

   Kepler problem is described by the potential energy

$$U(r) = -\frac{\alpha}{r}, \tag{6.41}$$

and the corresponding Lagrangian

$$L(\mathbf{r},\mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}. \tag{6.42}$$

For the case when the total energy E is negative,

$$-\frac{\alpha}{2r_0} < E < 0, \qquad r_0 = \frac{L_z^2}{\mu\alpha}, \tag{6.43}$$

where L_z is the angular momentum, the motion is described by an ellipse,

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi-\phi_0)}, \qquad e = \sqrt{1 + \frac{E}{(\alpha/2r_0)}}. \tag{6.44}$$

Perihelion is the point in the orbit of a planet when it is closest to the Sun. This corresponds to $\phi = \phi_0$. The precession of the perihelion is suitably defined in terms of the angular displacement $\Delta\phi$ of the perihelion during one revolution,

$$\Delta\phi = 2\left[\int_{r_{\min}}^{r_{\max}} d\phi\right] - 2\pi, \tag{6.45}$$

where

$$r_{\min} = \frac{r_0}{1+e} \tag{6.46}$$

is the perihelion, when the planet is closest to Sun, and

$$r_{\max} = \frac{r_0}{1-e} \tag{6.47}$$

is the aphelion, corresponding to $\phi = \phi_0 + \pi$, when the planet is farthest from Sun.

(a) For the Kepler problem derive the relation

$$d\phi = \frac{r_0\,dr}{r^2}\frac{1}{\sqrt{e^2 - \left(1 - \frac{r_0}{r}\right)^2}}. \tag{6.48}$$

Show that the precession of perihelion is zero for the Kepler problem.

(b) When a small correction

$$\delta U(r) = -\frac{\beta}{r^3} = \kappa\, U_0 \left(\frac{r_0}{r}\right)^3, \tag{6.49}$$

expressed in terms of dimensionless parameter κ using the relation $\beta = -\kappa U_0 r_0^3$, is added we have the perturbed potential energy

$$U(r) = -\frac{\alpha}{r} - \frac{\beta}{r^3} = -\frac{\alpha}{2r_0}\left[\frac{r_0}{r} + \kappa\left(\frac{r_0}{r}\right)^3\right]. \tag{6.50}$$

Show that the precession of the perihelion is no longer zero.

Chapter 7

Special relativity

7.1 Relativity principle

Problems

1. (**20 points.**) The relativity principle states that the laws of physics are invariant (or covariant) when observed using different coordinate systems. In special relativity we restrict these coordinate systems to be uniformly moving with respect to each other.

   (a) Linear: Spatial homogeneity, spatial isotropy, and temporal homogeneity, require the transformation to be linear. (We will skip this derivation. No submission needed.) Then, for simplicity, restricting to coordinate systems moving with respect to each other in a single direction, we can write

$$z' = A(v)\,z + B(v)\,t, \tag{7.1a}$$
$$t' = E(v)\,z + F(v)\,t. \tag{7.1b}$$

   We will refer to the respective frames as primed and unprimed.

   (b) Identity: An object P at rest in the primed frame, described by $z' = 0$, will be described in the unprimed frame as $z = vt$.

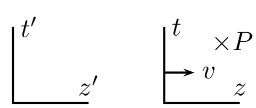

Figure 7.1: Identity.

   Using these in Eq. (7.1a), we have

$$0 = A(v)\,vt + B(v)\,t. \tag{7.2}$$

   This implies $B(v) = -vA(v)$. Thus, show that

$$z' = A(v)\,(z - vt), \tag{7.3a}$$
$$t' = E(v)\,z + F(v)\,t. \tag{7.3b}$$

   (c) Reversal: The descriptions of a process in the unprimed frame moving to the right with velocity v with respect to the primed should be identical to those made in the unprimed (with their axis flipped)

Figure 7.2: Reversal.

moving with velocity $-v$ with respect to the primed (with their axis flipped). This is equivalent to the requirement of isotropy in an one dimensional space.
That is, the transformation must be invariant under

$$z \to -z, \qquad z' \to -z', \qquad v \to -v. \tag{7.4}$$

This implies

$$-z' = A(-v)\,(-z + vt), \tag{7.5a}$$
$$t' = -E(-v)\,z + F(-v)\,t. \tag{7.5b}$$

Show that Eqs. (7.3a) and (7.5a) in conjunction imply

$$A(-v) = A(v). \tag{7.6}$$

Further, show that Eqs. (7.3b) and (7.5b) in conjunction implies

$$E(-v) = -E(v), \tag{7.7a}$$
$$F(-v) = F(v). \tag{7.7b}$$

(d) Reciprocity: The description of a process in the unprimed frame moving to the right with velocity v is identical to the description in the primed frame moving to the left.

Figure 7.3: Reciprocity.

That is, the transformation must be invariant under

$$(z,t) \to (z',t') \qquad (z',t') \to (z,t) \qquad v \to -v. \tag{7.8}$$

Show that this implies

$$z = A(-v)\,(z' + vt'), \tag{7.9a}$$
$$t = E(-v)\,z' + F(-v)\,t'. \tag{7.9b}$$

Show that Eqs. (7.3) and Eqs. (7.9) imply

$$E(v) = \frac{1}{v}\left[\frac{1}{A(v)} - A(v)\right], \tag{7.10a}$$
$$F(v) = A(v). \tag{7.10b}$$

Together, for arbitrary $A(v)$, the relativity principle allows the following transformations,

$$z' = A(v)\,(z - vt), \tag{7.11a}$$
$$t' = A(v)\left[\frac{1}{v}\left(\frac{1}{A(v)^2} - 1\right) z + t\right]. \tag{7.11b}$$

In Galilean relativity we require $t' = t$. Show that this is obtained with

$$A(v) = 1 \tag{7.12}$$

in Eqs. (7.11). This leads to the Galilean transformation

$$z' = z - vt, \tag{7.13a}$$
$$t' = t. \tag{7.13b}$$

In Einstein's special relativity the requirement is for a special speed c that is described identically by both the primed and unprimed frames. That is,

$$z = ct, \tag{7.14a}$$
$$z' = ct'. \tag{7.14b}$$

Show that Eqs. (7.14) when substituted in in Eqs. (7.11) leads to

$$A(v) = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}. \tag{7.15}$$

This corresponds to the Lorentz transformation

$$z' = A(v)(z - vt), \tag{7.16a}$$
$$t' = A(v)\left(-\frac{v}{c^2} z + t\right). \tag{7.16b}$$

This suggests that it should be possible to contrive additional solutions for $A(v)$ that respects the relativity principle, but with new physical requirements for the respective choice of $A(v)$. Construct one such transformation, which will not be used in grading.

7.2 Lorentz transformation

Problems

1. (**20 points.**) The Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \qquad \beta = \frac{v}{c}. \tag{7.17}$$

   (a) Evaluate γ for $v = 30\,\text{m/s}$ ($\sim 70\,\text{miles/hour}$).

   (b) Evaluate γ for $v = 3c/5$.

2. (**20 points.**) Lorentz transformation describing a boost in the x-direction is obtained using the matrix

$$L = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.18}$$

(a) Show that the determinant of the matrix L is 1.

(b) Determine L^{-1}.

3. (**20 points.**) Lorentz transformation (in one dimension) is given by

$$\Delta z' = \gamma(\Delta z - v\Delta t), \tag{7.19a}$$
$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c}\frac{\Delta z}{c}\right), \tag{7.19b}$$

where $\gamma = \sqrt{1 - v^2/c^2}$. Show that for

$$v \ll c \qquad \text{and} \qquad \frac{\Delta z}{\Delta t} \ll c \tag{7.20}$$

one obtains the Galilean transformation

$$\Delta z' = \Delta z - v\Delta t, \tag{7.21a}$$
$$\Delta t' = \Delta t. \tag{7.21b}$$

Note: For the case when Δz and Δt represent the change in position and time of a particle we could have v and $\Delta z/\Delta t$ to be identical.

4. (**20 points.**) How does the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(z - ct) = 0 \tag{7.22}$$

transform under the Lorentz transformtion

$$z' = \gamma z + \beta\gamma ct, \tag{7.23a}$$
$$ct' = \beta\gamma z + \gamma ct. \tag{7.23b}$$

Solution:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(a(z - ct)) = 0, \tag{7.24}$$

where $a = \sqrt{(1-\beta)/(1+\beta)}$.

5. (**20 points.**) Verify the following:

$$\mathrm{Tr}A = A_i{}^i. \tag{7.25a}$$
$$\det A = \varepsilon_{i_1 i_2 \ldots i_n} A^{i_1}{}_1 A^{i_2}{}_2 \ldots A^{i_n}{}_n \tag{7.25b}$$
$$= \frac{1}{n!}\varepsilon_{i_1 i_2 \ldots i_n}\varepsilon^{i'_1 i'_2 \ldots i'_n} A^{i_1}{}_{i'_1} A^{i_2}{}_{i'_2} \ldots A^{i_n}{}_{i'_n}, \tag{7.25c}$$

where n is the dimension of the matrix A.

6. (**20 points.**) Prove that any orthogonal matrix R satisfying

$$RR^T = 1 \tag{7.26}$$

in N-dimensions has $N(N-1)/2$ independent variables.

7. (**20 points.**) Lorentz transformation describing a boost in the x-direction, y-direction, and z-direction, are

$$L_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3\gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3\gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix}, \tag{7.27}$$

respectively. Transformation describing a rotation about the x-axis, y-axis, and z-axis, are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & 0 & -\sin\omega_1 & \cos\omega_1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\omega_2 & 0 & \cos\omega_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_3 & \sin\omega_3 & 0 \\ 0 & -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.28}$$

respectively. For infinitesimal transformations, $\beta_i = \delta\beta_i$ and $\omega_i = \delta\omega_i$ use the approximations

$$\gamma_i \sim 1, \qquad \cos\omega_i \sim 1, \qquad \sin\omega_i \sim \delta\omega_i, \tag{7.29}$$

to identify the generator for boosts $\mathbf{N}$, and the generator for rotations the angular momentum $\mathbf{J}$,

$$\mathbf{L} = \mathbf{1} + \delta\boldsymbol{\beta}\cdot\mathbf{N} \qquad \text{and} \qquad \mathbf{R} = \mathbf{1} + \delta\boldsymbol{\omega}\cdot\mathbf{J}, \tag{7.30}$$

respectively. Then derive

$$\Big[N_1, N_2\Big] = N_1N_2 - N_2N_1 = J_3. \tag{7.31}$$

This states that boosts in perpendicular direction leads to rotation. (To gain insight of the statement, calculate $[J_1, J_2]$ and interpret the result.)

(a) Is velocity addition commutative?

(b) Is velocity addition associative?

(c) Read a resource article (Wikipedia) on Wigner rotation.

8. (**20 points.**) (Based on Hughston and Tod's book.) Prove the following.

(a) If p^μ is a time-like vector and $p^\mu s_\mu = 0$ then s^μ is necessarily space-like.

(b) If p^μ and q^μ are both time-like vectors and $p^\mu q_\mu < 0$ then either both are future-pointing or both are past-pointing.

(c) If p^μ and q^μ are both light-like vectors and $p^\mu q_\mu = 0$ then p^μ and q^μ are proportional.

(d) If p^μ is a light-like vector and $p^\mu s_\mu = 0$ then s^μ is space-like or p^μ and q^μ are proportional.

(e) If u^α, v^α, and w^α, are time-like vectors with $u^\alpha v_\alpha < 0$ and $v^\alpha w_\alpha < 0$ then $v^\alpha w_\alpha < 0$.

9. (**20 points.**) Non-relativistic limits are obtained for $\beta \ll 1$ in relativistic formulae.

(a) Does Lorentz transformation recover Galilean transformation for $\beta \ll 1$?

(b) Does Lorentz transformation recover Galilean transformation for $\beta \ll 1$ and $c \to \infty$?

7.3 Geometry of Lorentz transformation

1. (**20 points.**) A four-vector in the context of Lorentz tranformation can be described using the notation

$$a^\alpha = (a^0, a^1, a^2, a^3). \tag{7.32}$$

Let

$$b^\alpha = (b^0, b^1, b^2, b^3) \tag{7.33}$$

be another four-vector. The scalar product between two Lorentz vectors is given by

$$a^\alpha b_\alpha = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3. \tag{7.34}$$

The square of the 'length' of the four-vector a^α is given by

$$a^\alpha a_\alpha, \tag{7.35}$$

which is not necessarily positive. The length of a four-vector is invariant, that is, it is independent of the Lorentz frame. If two Lorentz four-vectors are orthogonal they satisfy

$$a^\alpha b_\alpha = 0. \tag{7.36}$$

Orthogonality is an invariant concept.

(a) Determine the length of

$$p^\alpha = (5, 0, 0, 3), \tag{7.37}$$

where the numbers are in arbitrary units. Is it time-like, light-like, or space-like?

(b) Find a four-vector of the form

$$q^\alpha = (q^0, 0, 0, q^3) \tag{7.38}$$

that is perpendicular to p^α.

2. (**20 points.**) A hypothetical particle is observed by an inertial observer to be moving with non-uniform superluminal speed ($v > c$) at every instant of time from remote past to remote future. Draw a plausible world line of such a particle.

7.4 Poincaré (parallel) velocity addition formula

1. (**60 points.**) The Poincaré formula for the addition of (parallel) velocities is

$$v = \frac{v_a + v_b}{1 + \frac{v_a v_b}{c^2}}, \tag{7.39}$$

where v_a and v_b are velocities and c is speed of light in vacuum. Jerzy Kocik, from the department of Mathematics in SIUC, has invented a geometric diagram that allows one to visualize the Poincaré formula. (Refer [**?**].) An interactive applet for exploring velocity addition is available at Kocik's web page [**?**]. (For the following assume that the Poincaré formula holds for all speeds, subluminal ($v_i < c$), superluminal ($v_i > c$), and speed of light.)

(a) Analyse what is obtained if you add two subluminal speeds?

(b) Analyse what is obtained if you add a subluminal speed to speed of light?

(c) Analyse what is obtained if you add a subluminal speed to a superluminal speed?

(d) Analyse what is obtained if you add speed of light to another speed of light?

(e) Analyse what is obtained if you add a superluminal speed to speed of light?

(f) Analyse what is obtained if you add two superluminal speeds?

2. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is, $c = 1$,
$$v = \frac{v_a + v_b}{1 + v_a v_b}, \tag{7.40}$$
where v_a and v_b are velocities and c is speed of light in vacuum. Assuming that the Poincaré formula holds for all speeds, subluminal ($-1 < v_i < 1$), superluminal ($|v_i| > 1$), and speed of light, analyse what is obtained if you add a subluminal speed to a superluminal speed? That is, is the 'sum' subluminal or superluminal. Is the answer unique?

3. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is
$$v = \frac{v_a + v_b}{1 + \dfrac{v_a v_b}{c^2}}, \tag{7.41}$$
where v_a and v_b are velocities and c is speed of light in vacuum. (For the following assume that the Poincaré formula holds for all speeds, subluminal ($v_i < c$), superluminal ($v_i > c$), and speed of light.) Analyse what is obtained if you add a subluminal speed to a superluminal speed? That is, is the resultant speed subluminal or superluminal.
Hint: Analyse the case
$$\frac{v_a}{c} = -\frac{c}{v_b} \pm \delta, \tag{7.42}$$
for infinitely small $\delta > 0$.

4. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is, $c = 1$,
$$v = \frac{v_a + v_b}{1 + v_a v_b}, \tag{7.43}$$
where v_a and v_b are velocities and c is speed of light in vacuum. Assuming that the Poincaré formula holds for all speeds, subluminal ($-1 < v_i < 1$), superluminal ($|v_i| > 1$), and speed of light, analyse what is obtained if you add a speed to an infinitely large superluminal speed, that is, $v_b \to \infty$. Hint: Inversion.

5. (**30 points.**) Let
$$\tanh\theta = \beta, \tag{7.44}$$
where $\beta = v/c$. Addition of (parallel) velocities in terms of the parameter θ obeys the arithmatic addition
$$\theta = \theta_a + \theta_b. \tag{7.45}$$
   (a) Invert the expression in Eq. (7.44) to find the explicit form of θ in terms of β as a logarithm.
   (b) Show that Eq. (7.45) leads to the relation
$$\left(\frac{1+\beta}{1-\beta}\right) = \left(\frac{1+\beta_a}{1-\beta_a}\right)\left(\frac{1+\beta_b}{1-\beta_b}\right). \tag{7.46}$$
   (c) Using Eq. (7.46) derive the Poincaré formula for the addition of (parallel) velocities.

7.5 Kinematics

1. (**100 points.**) Relativisitic kinematics is constructed in terms of the proper time element ds, which remains unchanged under a Lorentz transformation,
$$-ds^2 = -c^2 dt^2 + d\mathbf{x}\cdot d\mathbf{x}. \tag{7.47}$$
Here $\mathbf{x}$ and t are the position and time of a particle. They are components of a vector under Lorentz transformation and together constitute the position four-vector
$$x^\alpha = (ct, \mathbf{x}). \tag{7.48}$$

(a) Velocity: The four-vector associated with velocity is constructed as

$$u^\alpha = c\frac{dx^\alpha}{ds}. \tag{7.49}$$

i. Using Eq. (7.47) deduce

$$\gamma ds = cdt, \quad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{7.50}$$

Then, show that

$$u^\alpha = (c\gamma, \mathbf{v}\gamma). \tag{7.51}$$

Here $\mathbf{v}$ is the velocity that we use in Newtonian physics.

ii. Further, show that

$$u^\alpha u_\alpha = -c^2. \tag{7.52}$$

Thus, conclude that the velocity four-vector is a time-like vector. What is the physical implication of this statement for a particle?

iii. Write down the form of the velocity four-vector in the rest frame of the particle?

(b) Momentum: Define momentum four-vector in terms of the mass m of the particle as

$$p^\alpha = mu^\alpha = (mc\gamma, m\mathbf{v}\gamma). \tag{7.53}$$

Connection with the physical quantities associated to a moving particle, the energy and momentum of the particle, is made by identifying (or defining)

$$p^\alpha = \left(\frac{E}{c}, \mathbf{p}\right), \tag{7.54}$$

which corresponds to the definitions

$$E = mc^2\gamma, \tag{7.55a}$$
$$\mathbf{p} = m\mathbf{v}\gamma, \tag{7.55b}$$

for energy and momentum, respectively. Discuss the non-relativistic limits of these quantities. In particular, using the approximation

$$\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots, \tag{7.56}$$

show that

$$E - mc^2 = \frac{1}{2}mv^2 + \ldots, \tag{7.57a}$$
$$\mathbf{p} = m\mathbf{v} + \ldots. \tag{7.57b}$$

Evaluate

$$p^\alpha p_\alpha = -m^2c^2. \tag{7.58}$$

Thus, derive the energy-momentum relation

$$E^2 - p^2c^2 = m^2c^4. \tag{7.59}$$

(c) Acceleration: The four-vector associated with acceleration is constructed as

$$a^\alpha = c\frac{du^\alpha}{ds}. \tag{7.60}$$

i. Show that
$$a^\alpha = \gamma \left(c\frac{d\gamma}{dt}, \mathbf{v}\frac{d\gamma}{dt} + \gamma \mathbf{a} \right), \tag{7.61}$$
where
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{7.62}$$
is the acceleration that we use in Newtonian physics.

ii. Starting from Eq. (7.52) and taking derivative with respect to proper time show that
$$u^\alpha a_\alpha = 0. \tag{7.63}$$
Thus, conclude that four-acceleration is space-like.

iii. Further, using the explicit form of $u^\alpha a_\alpha$ in Eq. (7.63) derive the identity
$$\frac{d\gamma}{dt} = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c^2} \right) \gamma^3. \tag{7.64}$$

iv. Show that
$$a^\alpha = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4, \mathbf{a}\gamma^2 + \frac{\mathbf{v}}{c}\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4 \right) \tag{7.65}$$

v. Write down the form of the acceleration four-vector in the rest frame ($\mathbf{v} = 0$) of the particle as $(0, \mathbf{a}_0)$, where
$$\mathbf{a}_0 = \mathbf{a}\big|_{\text{rest frame}} \tag{7.66}$$
is defined as the proper acceleration. Note that the proper acceleration is a Lorentz invariant quantity, that is, independent of which observer makes the measurement.

vi. Evaluate the following identities involving the proper acceleration
$$a^\alpha a_\alpha = \mathbf{a}_0\cdot\mathbf{a}_0 = \left[\mathbf{a}\cdot\mathbf{a} + \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c} \right)^2 \gamma^2 \right] \gamma^4 = \left[\mathbf{a}\cdot\mathbf{a} - \left(\frac{\mathbf{v}\times\mathbf{a}}{c} \right)^2 \right] \gamma^6. \tag{7.67}$$

vii. In a particular frame, if $\mathbf{v}\,||\,\mathbf{a}$ (corresponding to linear motion), deduce
$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^3. \tag{7.68}$$
And, in a particular frame, if $\mathbf{v} \perp \mathbf{a}$ (corresponding to circular motion), deduce
$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^2. \tag{7.69}$$

(d) Force: The force four-vector is defined as
$$f^\alpha = c\frac{dp^\alpha}{ds} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \mathbf{F}\gamma \right), \tag{7.70}$$
where the force $\mathbf{F}$, identified (or defined) as
$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{7.71}$$
is the force in Newtonian physics. Starting from Eq. (7.58) derive the relation
$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} \tag{7.72}$$
which is the power output or the rate of work done by the force $\mathbf{F}$ on the particle.

(e) Equations of motion: The relativistic generalization of Newton's laws are

$$f^\alpha = ma^\alpha. \tag{7.73}$$

Show that these involve the relations, using the definitions of energy and momentum in Eqs. (7.55),

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}\gamma + m\mathbf{v}\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\gamma^3, \tag{7.74a}$$

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} = m\mathbf{v}\cdot\mathbf{a}\gamma^3. \tag{7.74b}$$

Discuss the non-relativistic limits of the equations of motion.

2. (**20 points.**) Lorentz transformation relates the energy E and momentum $\mathbf{p}$ of a particle when measured in different frames. For example, for the special case when the relative velocity and the velocity of the particle are parallel we have

$$\begin{pmatrix} E'/c \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ p \end{pmatrix}. \tag{7.75}$$

Photons are massless spin 1 particles whose energy and momentum are $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, such that $\omega = kc$. Thus, derive the relativistic Doppler effect formula

$$\omega' = \omega\sqrt{\frac{1+\beta}{1-\beta}}. \tag{7.76}$$

3. (**20 points.**) Neutral π meson decays into two photons. That is,

$$\pi^0 \to \gamma_1 + \gamma_2. \tag{7.77}$$

Energy-momentum conservation for the decay in the laboratory frame, in which the meson is not necessarily at rest, is given by

$$p_\pi^\alpha = p_1^\alpha + p_2^\alpha. \tag{7.78}$$

Or, more specifically,

$$\left(\frac{E_\pi}{c}, \mathbf{p}\right) = \left(\frac{E_1}{c}, \mathbf{p}_1\right) + \left(\frac{E_2}{c}, \mathbf{p}_2\right), \tag{7.79}$$

where E_π and $\mathbf{p}$ are the energy and momentum of neutral π meson, and E_i's and $\mathbf{p}_i$'s are the energies and momentums of the photons. Thus, derive the relation

$$m_\pi^2 c^4 = 2E_1E_2(1-\cos\theta), \tag{7.80}$$

where m_π is the mass of neutral π meson, and θ is the angle between the directions of $\mathbf{p}_1$ and $\mathbf{p}_2$.

4. (**20 points.**) Using Maxwell's equations we can show that a monochromatic electromagnetic wave has the electromagnetic energy density U and electromagnetic momentum density $\mathbf{G}$ given by

$$U = \frac{1}{2}\varepsilon_0^2E^2 + \frac{1}{2}\mu_0^2H^2 = \varepsilon_0^2E^2 = \mu_0^2H^2, \tag{7.81}$$

$$\mathbf{G} = \frac{\mathbf{E}\times\mathbf{H}}{c^2} = \hat{\mathbf{k}}\frac{U}{c}. \tag{7.82}$$

Thus, the energy momentum four-vector for a monochromatic electromagnetic wave is given by

$$p^\alpha = \left(\frac{U}{c}, \mathbf{G}\right) = \frac{U}{c}\left(1, \hat{\mathbf{k}}\right). \tag{7.83}$$

Note: Complete this!

5. (**20 points.**) Length contracts and time dilates. That is,
$$L = \frac{L_0}{\gamma}, \qquad T = T_0\gamma, \tag{7.84}$$
where L_0 and T_0 are proper length and proper time. Similarly, show that (for $\mathbf{v} \,||\, \mathbf{a}$)
$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^3}, \tag{7.85}$$
where $|\mathbf{a}_0|$ is the proper acceleration measured in the instantaneaous rest frame of the particle. Further, for $\mathbf{v} \perp \mathbf{a}$ show that
$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^2}. \tag{7.86}$$

6. (**20 points.**) Time dilates. That is,
$$T = T_0\gamma, \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{7.87}$$
where T_0 is the proper time measured in the instantaneous rest frame of the clock measuring T_0 and T is the time measured by a clock moving with velocity v relative to the clock measuring proper time. Similarly, show that (for $\mathbf{v} \,||\, \mathbf{a}$)
$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^3}, \tag{7.88}$$
where $|\mathbf{a}_0|$ is the proper acceleration measured in the instantaneous rest frame of the particle. Derive the equation for the trajectory of a particle moving in a straight line (along the z axis) with constant proper acceleration, after starting from rest from the point $z = c^2/|\mathbf{a}_0|$ at time $t = 0$.

7.6 Dynamics

7.6.1 Charge particle in a uniform magnetic field: Circular motion

1. (**20 points.**) A relativisitic particle in a uniform magnetic field is described by the equations
$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{7.89a}$$
$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{7.89b}$$
where
$$E = mc^2\gamma, \tag{7.90a}$$
$$\mathbf{p} = m\mathbf{v}\gamma, \tag{7.90b}$$
and
$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{7.91}$$
Show that
$$\frac{d\gamma}{dt} = 0. \tag{7.92}$$
Then, derive
$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \boldsymbol{\omega}_c, \tag{7.93}$$
where
$$\boldsymbol{\omega}_c = \frac{q\mathbf{B}}{m\gamma}. \tag{7.94}$$
Compare this relativistic motion to the associated non-relativistic motion.

2. (**20 points.**) If the motion of a non-relativistic particle is such that it does not change the kinetic energy of the particle, we have

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = 0. \tag{7.95}$$

Show that this imples

$$\mathbf{v}\cdot\mathbf{a} = 0. \tag{7.96}$$

This is achieved when the acceleration $a = 0$ or in the case of uniform circular motion. Starting from Eq. (7.96) show that the relativistic generalization of kinetic energy $E = mc^2\gamma$ is also conserved, that is,

$$\frac{d}{dt}(mc^2\gamma) = 0. \tag{7.97}$$

Observe that

$$\boldsymbol{\beta}\cdot\mathbf{a} = \frac{d}{dt}\left(\frac{\beta^2}{2}\right) = -\frac{1}{2}\frac{d}{dt}\frac{1}{\gamma^2} = \frac{1}{\gamma^3}\frac{d\gamma}{dt}. \tag{7.98}$$

7.6.2 Charge particle in a uniform electric field: Hyperbolic motion

1. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v}, \tag{7.99a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{7.99b}$$

where

$$E = mc^2\gamma, \tag{7.100a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{7.100b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{7.101}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{7.102}$$

and initial conditions

$$\mathbf{v}(0) = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{7.103a}$$

$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}. \tag{7.103b}$$

(a) In terms of the definition

$$\boldsymbol{\omega}_0 = \frac{1}{c}\frac{q\mathbf{E}}{m}, \tag{7.104}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0\cdot\boldsymbol{\beta} \tag{7.105}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{7.106}$$

(b) Since the particle starts from rest show that we have

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t. \tag{7.107}$$

For our configuration this implies

$$\beta_x = 0, \tag{7.108a}$$
$$\beta_y \gamma = \omega_0 t, \tag{7.108b}$$
$$\beta_z = 0. \tag{7.108c}$$

Further, deduce

$$\beta_y = \frac{\omega_0 t}{\sqrt{1+\omega_0^2 t^2}}. \tag{7.109}$$

Integrate again and use the initial condition to show that the motion is described by

$$y - y_0 = \frac{c}{\bar{\omega}_0}\left[\sqrt{1+\bar{\omega}_0^2 t^2} - 1\right]. \tag{7.110}$$

Rewrite the solution in the form

$$\left(y - y_0 + \frac{c}{\omega_0}\right)^2 - c^2 t^2 = \frac{c^2}{\omega_0^2}. \tag{7.111}$$

This represents a hyperbola passing through $y = y_0$ at $t = 0$. If we choose the initial position $y_0 = c/\omega_0$ we have

$$y^2 - c^2 t^2 = y_0^2. \tag{7.112}$$

(c) The (constant) proper acceleration associated with this motion is

$$\alpha = \omega_0 c = \frac{c^2}{y_0}. \tag{7.113}$$

A Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$y - y_0 = \frac{1}{2}\alpha t^2. \tag{7.114}$$

Show that the hyperbolic curve

$$y = y_0\sqrt{1+\frac{c^2t^2}{y_0^2}} \tag{7.115}$$

in regions that satisfy

$$\omega_0 t \ll 1 \tag{7.116}$$

is approximately the parabolic curve

$$y = y_0 + \frac{1}{2}\alpha t^2 + \dots. \tag{7.117}$$

2. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by equation of a hyperbola

$$z^2 - c^2 t^2 = z_0^2, \qquad z_0 = \frac{c^2}{\alpha}. \tag{7.118}$$

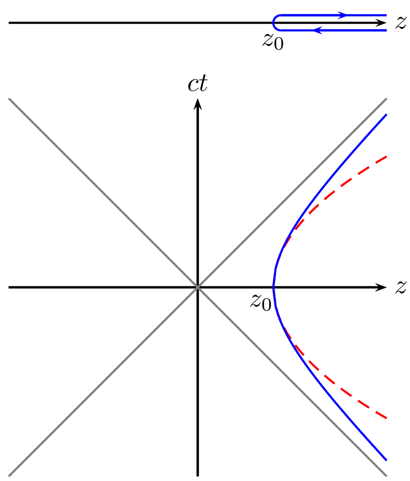

Figure 7.4: Problem 2

(a) This represents the world-line of a particle thrown from $z > z_0$ at $t < 0$ towards $z = z_0$ in region of constant (proper) acceleration α as described by the bold (blue) curve in the space-time diagram in Figure 2. In contrast a Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$z - z_0 = \frac{1}{2}\alpha t^2 \tag{7.119}$$

as described by the dashed (red) curve in the space-time diagram in Figure 2. Show that the hyperbolic curve

$$z = z_0\sqrt{1 + \frac{c^2t^2}{z_0^2}} \tag{7.120}$$

in regions that satisfy

$$t \ll \frac{c}{\alpha} \tag{7.121}$$

is approximately the parabolic curve

$$z = z_0 + \frac{1}{2}\alpha t^2 + \ldots. \tag{7.122}$$

(b) Recognize that the proper acceleration α does not have an upper bound.

(c) A large acceleration is achieved by taking an above turn while moving very fast. Thus, turning around while moving close to the speed of light c should achieve the highest acceleration. Show that $\alpha \to \infty$ corresponding to $z_0 \to 0$ represents this scenario. What is the equation of motion of a particle moving with infinite proper acceleration. To gain insight, plot world-lines of particles moving with $\alpha = c^2/z_0$, $\alpha = 10c^2/z_0$, and $\alpha = 100c^2/z_0$.

3. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by the equation of a hyperbola

$$z^2 - c^2t^2 = z_0^2, \qquad z_0 = \frac{c^2}{\alpha}. \tag{7.123}$$

This is the motion of a particle 'dropped' from $z = z_0$ at $t = 0$ in region of constant (proper) acceleration. See Figure 3. Using geometric (diagrammatic) arguments might be easiest to answer the following.

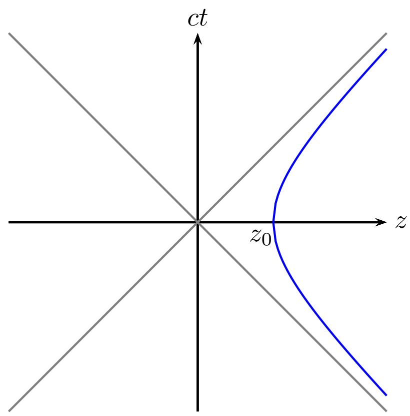

Figure 7.5: Problem 3

(a) Will a photon dispatched to 'chase' this particle at $t = 0$ from $z = 0$ ever catch up with it? If yes, when and where does it catch up?

(b) Will a photon dispatched to 'chase' this particle at $t = 0$ from $0 < z < z_0$ ever catch up with it? If yes, when and where does it catch up?

(c) Will a photon dispatched to 'chase' this particle, at $t = 0$ from $z < 0$ ever catch up with it? If yes, when and where does it catch up?

What are the implications for the observable part of our universe from this analysis?

4. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_q(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{7.124}$$

This is the motion of a particle that comes to existance at $z_q = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Assume you are positioned at the origin. If the particle is a source of light (imagine a flash light) at what time will the light first reach you at the origin? Where is the particle when this happens?

5. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_2(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{7.125}$$

This is the motion of a particle that comes to existance at $z_2 = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_2(0) = 0$ and $y_2(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Another particle is at rest at z_1

$$z_1(t) = z_1, \tag{7.126}$$

such that $0 < z_1 < z_0$. Assume that both particles emit photons continuously.

(a) At what time do photons emitted by 2 first reach 1? Where is particle 2 when this happens?

(b) At what time is the last photon that reaches 2 emitted by 1? Where is particle 2 when this happens?

(c) Do all the photons emitted by 1 reach 2?

(d) Do all the photons emitted by 2 reach 1?

6. (**20 points.**) The path of a relativistic particle 1 moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_1(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{7.127}$$

This is the motion of a particle that comes to existance at $z_1 = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Consider another relavistic particle 2 undergoing hyperbolic motion given by

$$z_2(t) = -\sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{7.128}$$

This is the motion of a particle that comes to existance at $z_2 = -\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = -z_0$, and then returns back to negative infinity. The world-line of particle 1 is the blue curve in Figure 6, and the world-line of particle 2 is the red curve in Figure 6. Using geometric (diagrammatic) arguments might be easiest to answer the following. Imagine the particles are sources of light (imagine a flash light pointing towards origin).

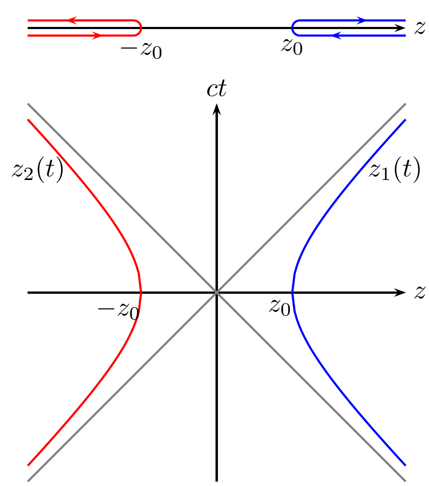

Figure 7.6: Problem 6

(a) At what time will the light from particle 1 first reach particle 2? Where are the particles when this happens?

(b) At what time will the light from particle 2 first reach particle 1? Where are the particles when this happens?

(c) Can the particles communicate with each other?

(d) Can the particles ever detect the presence of the other? In other words, can one particle be aware of the existence of the other? What can you deduce about the observable part of our universe from this analysis?

7. (**20 points.**) Two masses (one heavier than the other) move with constant proper acceleration α, after they are dropped from position $x_0 = c^2/\alpha$. Does the time taken to fall a given distance depend on mass?

Recall that Aristotle (384-322 BC) presumed that the time taken to fall a given distance depended on mass. Galileo (1564-1642) argued, based on a famous thought experiment (refer Wikipedia) that the time taken to fall a given distance is independent of mass.

(a) Consider an electron and a proton connected by a hypothetical string. What is the tension in the string when they move in a uniform electric field (which leads to proper acceleration). We will have to dictate how the distance between them changes.

(b) What about charges of different masses in an electric field?

(c) What about a hydrogen atom? How does electrostatic energy associated to the hydrogen atom fall?

(d) Do these considerations involve a Poincare stress?

Keywords: Trouton-Noble experiment, Laue current, 4/3 problem.
NOTE: This problem needs thought and scrutiny!

7.6.3 Charge particle in a uniform electric field with an initial velocity normal to electric field: Hyperbolic motion

1. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{7.129a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{7.129b}$$

where

$$E = mc^2\gamma, \tag{7.130a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{7.130b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{7.131}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{7.132}$$

and initial conditions

$$\mathbf{v}(0) = v_0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{7.133a}$$

$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}. \tag{7.133b}$$

We will use the associated definitions $\boldsymbol{\beta}_0 = \mathbf{v}(0)/c$ and $\boldsymbol{\gamma}_0 = 1/\sqrt{1-\beta_0^2}$.

(a) In terms of the definition

$$\boldsymbol{\omega}_0 = \frac{1}{c}\frac{q\mathbf{E}}{m}, \tag{7.134}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0 \cdot \boldsymbol{\beta} \tag{7.135}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{7.136}$$

(b) For our configuration show that

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t - \beta_0\gamma_0\hat{\mathbf{x}}, \tag{7.137}$$

such that

$$\beta_x\gamma = -\beta_0\gamma_0, \tag{7.138a}$$
$$\beta_y\gamma = \omega_0 t, \tag{7.138b}$$
$$\beta_z\gamma = 0. \tag{7.138c}$$

Using $\beta_z\gamma = 0$, learn that

$$\frac{\beta_z^2}{1-\beta_x^2-\beta_y^2-\beta_z^2} = 0 \tag{7.139}$$

and in conjunction with $\beta_x\gamma = -\beta_0\gamma_0$ deduce that

$$\beta_z = 0 \tag{7.140}$$

and

$$\frac{\beta_x^2}{\beta_0^2} + \beta_y^2 = 1. \tag{7.141}$$

Thus, deduce

$$\gamma^2 = \omega_0^2 t^2 + \gamma_0^2 \tag{7.142}$$

and

$$\beta_x^2 + \beta_y^2 = \beta_0^2 + \frac{\beta_y^2}{\gamma_0^2}. \tag{7.143}$$

Further, deduce

$$\beta_y = \frac{\bar{\omega}_0 t}{\sqrt{1+\bar{\omega}_0^2 t^2}} \tag{7.144}$$

and

$$\beta_x = \frac{\beta_0}{\sqrt{1+\bar{\omega}_0^2 t^2}}, \tag{7.145}$$

where

$$\bar{\omega}_0 = \frac{\omega_0}{\gamma_0}. \tag{7.146}$$

Integrate again and use the initial condition to show that the motion is described by

$$y - y_0 = \frac{c}{\bar{\omega}_0}\left[\sqrt{1+\bar{\omega}_0^2 t^2} - 1\right], \tag{7.147a}$$
$$x - x_0 = \frac{v_0}{\bar{\omega}_0}\sinh^{-1}\bar{\omega}_0 t, \tag{7.147b}$$

and $z = 0$.

(c) Show that for $v_0 = 0$ we reproduce the solution for a particle starting from rest. Next, for

$$\bar{\omega}_0 t \ll 1 \tag{7.148}$$

and

$$\alpha = \bar{\omega}_0 c \tag{7.149}$$

obtain the non-relativistic limits,

$$y - y_0 = \frac{1}{2}\alpha t^2, \tag{7.150a}$$
$$x - x_0 = v_0 t. \tag{7.150b}$$

Hint: Recall the series expansion

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) = x + \dots. \tag{7.151}$$

7.7 Electrodynamics

Problems

1. (**20 points.**) In terms of the four-vector potential
$$cA^\mu = (\phi, c\mathbf{A}) \tag{7.152}$$
the Maxwell field tensor $F_{\mu\nu}$ is defined as
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{7.153}$$
and the corresponding dual tensor is defined as
$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. \tag{7.154}$$
Derive the following relations, which involve quantities that remain invariant under Lorentz transformations.
$$c^2F^{\mu\nu}F_{\mu\nu} = 2(c^2B^2 - E^2). \tag{7.155a}$$
$$c^2\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = 2(E^2 - c^2B^2). \tag{7.155b}$$
$$cF^{\mu\nu}\tilde{F}_{\mu\nu} = -4\,\mathbf{B}\cdot\mathbf{E}. \tag{7.155c}$$

2. (**100 points.**) Eigenvalues of the energy momentum tensor. (We choose $c = 1$, which is easily undone by replacing $\mathbf{E} \to \frac{1}{c}\mathbf{E}$ everywhere.)

   (a) Using
$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{7.156}$$
   and
$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{7.157}$$
   evaluate the following:

   i. $F^{\mu\lambda}F_{\lambda\nu}$
   ii. $\tilde{F}^{\mu\lambda}\tilde{F}_{\lambda\nu}$
   iii. Then, derive
$$F^{\mu\lambda}\tilde{F}_{\lambda\nu} = \delta^\mu{}_\nu\mathbf{E}\cdot\mathbf{B}, \tag{7.158a}$$
$$\tilde{F}^{\mu\lambda}\tilde{F}_{\lambda\nu} - F^{\mu\lambda}F_{\lambda\nu} = \delta^\mu{}_\nu(B^2 - E^2). \tag{7.158b}$$

   (b) Define
$$2f = (B^2 - E^2) \quad \text{and} \quad g = \mathbf{E}\cdot\mathbf{B}. \tag{7.159}$$
   Thus, construct matrix (or dyadic) equations
$$\mathbf{F}\cdot\tilde{\mathbf{F}} = g\mathbf{1}, \tag{7.160a}$$
$$\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}} - \mathbf{F}\cdot\mathbf{F} = 2f\mathbf{1}, \tag{7.160b}$$
   in terms of matrices (or dyadics) $\mathbf{F}$ and $\tilde{\mathbf{F}}$.

(c) Show that the eigenvalues λ of the field tensor $\mathbf{F}$ satisfy the quartic equation

$$\lambda^4 + 2f\lambda^2 - g^2 = 0. \tag{7.161}$$

(d) Evaluate the eigenvalues to be $\pm\lambda_1$ and $\pm\lambda_2$ where

$$\lambda_1 = \sqrt{-f - \sqrt{f^2+g^2}} = \frac{i}{\sqrt{2}}\left[\sqrt{f+ig} + \sqrt{f-ig}\right], \tag{7.162}$$

$$\lambda_2 = \sqrt{-f + \sqrt{f^2+g^2}} = \frac{i}{\sqrt{2}}\left[\sqrt{f+ig} - \sqrt{f-ig}\right]. \tag{7.163}$$

(e) Show that

i. if $B^2 - E^2 = 0$, then the eigenvalues are $\pm\sqrt{g}$ and $\pm i\sqrt{g}$.

ii. if $\mathbf{B}\cdot\mathbf{E} = 0$, then the eigenvalues are 0, 0, and $\pm\sqrt{2f}$.

(f) Prove the following:

i. There is no Lorentz transformation connecting two reference frames such that the field is purely magnetic in origin in one and purely electric in origin in the other.

ii. If $B^2 - E^2 > 0$ in a frame, then there exists a frame in which the field is purely magnetic.

iii. If $B^2 - E^2 < 0$ in a frame, then there exists a frame in which the field is purely electric.

iv. If $B^2 - E^2 = 0$ in a frame, then there exists a frame in which

— $\mathbf{B}$ is perpendicular to $\mathbf{E}$, if $\mathbf{B}\cdot\mathbf{E} = 0$.

— $\mathbf{B}$ is parallel to $\mathbf{E}$, if $\mathbf{B}\cdot\mathbf{E} > 0$.

— $\mathbf{B}$ is antiparallel to $\mathbf{E}$, if $\mathbf{B}\cdot\mathbf{E} < 0$.

3. (**40 points.**) The electric and magnetic fields transform under a Lorentz transformation (for boost in z direction) as

$$E'_x(\mathbf{r}',t') = \gamma\, E_x(\mathbf{r},t) + \beta\gamma\, cB_y(\mathbf{r},t), \tag{7.164a}$$

$$cB'_y(\mathbf{r}',t') = \beta\gamma\, E_x(\mathbf{r},t) + \gamma\, cB_y(\mathbf{r},t), \tag{7.164b}$$

$$E'_z(\mathbf{r}',t') = E_z(\mathbf{r},t) \tag{7.164c}$$

$$cB'_x(\mathbf{r}',t') = \gamma\, cB_x(\mathbf{r},t) - \beta\gamma\, E_y(\mathbf{r},t), \tag{7.165a}$$

$$E'_y(\mathbf{r}',t') = -\beta\gamma\, cB_x(\mathbf{r},t) + \gamma\, E_y(\mathbf{r},t), \tag{7.165b}$$

$$cB'_z(\mathbf{r}',t') = cB_z(\mathbf{r},t), \tag{7.165c}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. The transformed values of the coordinates and the fields are distinguished by a prime. Derive the invariance properties

$$\mathbf{E}'(\mathbf{r}',t')\cdot\mathbf{B}'(\mathbf{r}',t') = \mathbf{E}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{r},t) \tag{7.166}$$

and

$$\mathbf{E}'(\mathbf{r}',t')^2 - c^2\mathbf{B}'(\mathbf{r}',t')^2 = \mathbf{E}(\mathbf{r},t)^2 - c^2\mathbf{B}(\mathbf{r},t)^2. \tag{7.167}$$

4. (**20 points.**) Let an infinitely thin plate occupying the $y = 0$ plane consist of a uniform charge density flowing in the $\hat{\mathbf{x}}$ direction described by drift velocity $\beta_d = v/c$.

(a) Show that the electric and magnetic field for this configuration is given by

$$\mathbf{E} = \eta(y)\,\hat{\mathbf{y}}\,\frac{\sigma}{2\varepsilon_0}, \tag{7.168a}$$

$$c\mathbf{B} = \eta(y)\,\hat{\mathbf{z}}\,\beta_d E, \tag{7.168b}$$

where

$$\eta(y) = \begin{cases} 1, & y > 0, \\ -1, & y < 0. \end{cases} \tag{7.169}$$

Thus, we have

$$cB = \beta_d E. \tag{7.170}$$

Recall that the motion of a point charge in this field configuration is a cycloid,

$$x(t) - v_q t = R \sin \omega_c t, \tag{7.171a}$$

$$y(t) - R = R \cos \omega_c t, \tag{7.171b}$$

that satisfies

$$\big[x(t) - v_q t\big]^2 + \big[y(t) - R\big]^2 = R^2, \tag{7.172}$$

where

$$\omega_c = \frac{qB}{m}, \quad v_q = \frac{E}{B} \quad \text{and} \quad R = \frac{v_q}{\omega_c}. \tag{7.173}$$

(b) Show that under a Lorentz transformation (for boost in x direction) the electric and magnetic fields transform as

$$\mathbf{E}' = \hat{\mathbf{y}}\, E', \tag{7.174a}$$

$$c\mathbf{B}' = \hat{\mathbf{z}}\, B'\eta(y), \tag{7.174b}$$

where

$$E' = \gamma(E - \beta cB), \tag{7.175a}$$

$$cB' = \gamma(cB - \beta E). \tag{7.175b}$$

Verify that

$$E'^2 - (cB)'^2 = E^2 - (cB)^2 \tag{7.176}$$

and

$$\mathbf{E}' \cdot \mathbf{B}' = \mathbf{E} \cdot \mathbf{B} = 0. \tag{7.177}$$

(c) Verify that for $\beta = \beta_d < 1$ we have $B' = 0$ and $E' = E/\gamma_d$. Investigate what happens to the radius R and the pitch of the cycloid $2\pi R$ in this case.

(d) Note that for $\beta = E/(cB) > 1$ we have $B' = B/\gamma$ and $E' = 0$. Investigate what happens.