# Homework No. 12 (2024 Spring)

## PHYS 510: CLASSICAL MECHANICS

*School of Physics and Applied Physics, Southern Illinois University–Carbondale*

Due date: Friday, 2024 May 3, 4.30pm

0. Problem 4 is optional.

1. (**20 points.**) The relativity principle states that the laws of physics are invariant (or covariant) when observed using different coordinate systems. In special relativity we restrict these coordinate systems to be uniformly moving with respect to each other. Let $z = z' = 0$ at $t = 0$.

   (a) Linear: Spatial homogeneity, spatial isotropy, and temporal homogeneity, require the transformation to be linear. (We will skip this derivation.) Then, for simplicity, restricting to coordinate systems moving with respect to each other in a single direction, we can write

   $$z' = A(v)\, z + B(v)\, t, \tag{1a}$$
   $$t' = E(v)\, z + F(v)\, t. \tag{1b}$$

   We will refer to the respective frames as primed and unprimed.

   (b) Identity: An object $P$ at rest in the primed frame, described by $z' = 0$, will be described in the unprimed frame as $z = vt$.

   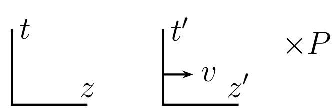
   

   Figure 1: Identity.

   Using these in Eq. (1a), we have

   $$0 = A(v)\, vt + B(v)\, t. \tag{2}$$

   This implies $B(v) = -vA(v)$. Thus, show that

   $$z' = A(v)\,(z - vt), \tag{3a}$$
   $$t' = E(v)\, z + F(v)\, t. \tag{3b}$$


Figure 2: Reversal.

(c) Reversal: The descriptions of a process in the unprimed frame moving to the right with velocity $v$ with respect to the primed should be identical to those made in the unprimed (with their axis flipped) moving with velocity $-v$ with respect to the primed (with their axis flipped). This is equivalent to the requirement of isotropy in an one dimensional space.

That is, the transformation must be invariant under

$$z \to -z, \qquad z' \to -z', \qquad v \to -v. \tag{4}$$

This implies

$$-z' = A(-v)\,(-z + vt), \tag{5a}$$

$$t' = -E(-v)\,z + F(-v)\,t. \tag{5b}$$

Show that Eqs. (3a) and (5a) in conjunction imply

$$A(-v) = A(v). \tag{6}$$

Further, show that Eqs. (3b) and (5b) in conjunction implies

$$E(-v) = -E(v), \tag{7a}$$

$$F(-v) = F(v). \tag{7b}$$

(d) Reciprocity: The description of a process in the unprimed frame moving to the right with velocity $v$ is identical to the description in the primed frame moving to the left.


Figure 3: Reciprocity.

That is, the transformation must be invariant under

$$(z, t) \to (z', t') \qquad (z', t') \to (z, t) \qquad v \to -v. \tag{8}$$

Show that this implies

$$z = A(-v)\,(z' + vt'), \tag{9a}$$
$$t = E(-v)\,z' + F(-v)\,t'. \tag{9b}$$

Show that Eqs. (3) and Eqs. (9) imply

$$E(v) = \frac{1}{v}\left[\frac{1}{A(v)} - A(v)\right], \tag{10a}$$
$$F(v) = A(v). \tag{10b}$$

(e) Together, for arbitrary $A(v)$, show that the relativity principle allows the following transformations,

$$z' = A(v)\,(z - vt), \tag{11a}$$
$$t' = A(v)\left[\frac{1}{v}\left(\frac{1}{A(v)^2} - 1\right)z + t\right]. \tag{11b}$$

i. In Galilean relativity we require $t' = t$. Show that this is obtained with

$$A(v) = 1 \tag{12}$$

in Eqs. (11). This leads to the Galilean transformation

$$z' = z - vt, \tag{13a}$$
$$t' = t. \tag{13b}$$

ii. In Einstein's special relativity the requirement is for a special speed $c$ that is described identically by both the primed and unprimed frames. That is,

$$z = ct, \tag{14a}$$
$$z' = ct'. \tag{14b}$$

Show that Eqs. (14) when substituted in in Eqs. (11) leads to

$$A(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{15}$$

This corresponds to the Lorentz transformation

$$z' = A(v)(z - vt), \tag{16a}$$
$$t' = A(v)\left(-\frac{v}{c^2}z + t\right). \tag{16b}$$

iii. This suggests that it should be possible to contrive additional solutions for $A(v)$ that respects the relativity principle, but with new physical requirements for the respective choice of $A(v)$. Construct one such transformation. In particular, investigate modifications of Eqs. (14) that donot change the current experimental observations. The response to this part of the question will not be used for assessment.

2. (**20 points.**) (Based on Hughston and Tod's book.) Prove the following.

   (a) If $p^\mu$ is a time-like vector and $p^\mu s_\mu = 0$ then $s^\mu$ is necessarily space-like.

   (b) If $p^\mu$ and $q^\mu$ are both time-like vectors and $p^\mu q_\mu < 0$ then either both are future-pointing or both are past-pointing.

   (c) If $p^\mu$ and $q^\mu$ are both light-like vectors and $p^\mu q_\mu = 0$ then $p^\mu$ and $q^\mu$ are proportional.

   (d) If $p^\mu$ is a light-like vector and $p^\mu s_\mu = 0$ then $s^\mu$ is space-like or $p^\mu$ and $q^\mu$ are proportional.

   (e) If $u^\alpha$, $v^\alpha$, and $w^\alpha$, are time-like vectors with $u^\alpha v_\alpha < 0$ and $v^\alpha w_\alpha < 0$ then $v^\alpha w_\alpha < 0$.

3. (**30 points.**) Let

$$\tanh\theta = \beta, \tag{17}$$

where $\beta = v/c$. Addition of (parallel) velocities in terms of the parameter $\theta$ obeys the arithmatic addition

$$\theta = \theta_a + \theta_b. \tag{18}$$

   (a) Invert the expression in Eq. (17) to find the explicit form of $\theta$ in terms of $\beta$ as a logarithm.

   (b) Show that Eq. (18) leads to the relation

$$\left(\frac{1+\beta}{1-\beta}\right) = \left(\frac{1+\beta_a}{1-\beta_a}\right)\left(\frac{1+\beta_b}{1-\beta_b}\right). \tag{19}$$

   (c) Using Eq. (19) derive the Poincaré formula for the addition of (parallel) velocities.

4. (**20 points.**) Relativisitic kinematics is constructed in terms of the proper time element $ds$, which remains unchanged under a Lorentz transformation,

$$-ds^2 = -c^2dt^2 + d\mathbf{x}\cdot d\mathbf{x}. \tag{20}$$

Here $\mathbf{x}$ and $t$ are the position and time of a particle. They are components of a vector under Lorentz transformation and together constitute the position four-vector

$$x^\alpha = (ct, \mathbf{x}). \tag{21}$$

   (a) Velocity: The four-vector associated with velocity is constructed as

$$u^\alpha = c\frac{dx^\alpha}{ds}. \tag{22}$$

i. Using Eq. (20) deduce

$$\gamma ds = cdt, \quad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{23}$$

Then, show that

$$u^\alpha = (c\gamma, \mathbf{v}\gamma). \tag{24}$$

Here $\mathbf{v}$ is the velocity that we use in Newtonian physics.

ii. Further, show that

$$u^\alpha u_\alpha = -c^2. \tag{25}$$

Thus, conclude that the velocity four-vector is a time-like vector. What is the physical implication of this statement for a particle?

iii. Write down the form of the velocity four-vector in the rest frame of the particle?

(b) Momentum: Define momentum four-vector in terms of the mass $m$ of the particle as

$$p^\alpha = mu^\alpha = (mc\gamma, m\mathbf{v}\gamma). \tag{26}$$

Connection with the physical quantities associated to a moving particle, the energy and momentum of the particle, is made by identifying (or defining)

$$p^\alpha = \left(\frac{E}{c}, \mathbf{p}\right), \tag{27}$$

which corresponds to the definitions

$$E = mc^2\gamma, \tag{28a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{28b}$$

for energy and momentum, respectively. Discuss the non-relativistic limits of these quantities. In particular, using the approximation

$$\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots, \tag{29}$$

show that

$$E - mc^2 = \frac{1}{2}mv^2 + \ldots, \tag{30a}$$

$$\mathbf{p} = m\mathbf{v} + \ldots. \tag{30b}$$

Evaluate

$$p^\alpha p_\alpha = -m^2c^2. \tag{31}$$

Thus, derive the energy-momentum relation

$$E^2 - p^2c^2 = m^2c^4. \tag{32}$$

(c) Acceleration: The four-vector associated with acceleration is constructed as

$$a^\alpha = c\frac{du^\alpha}{ds}. \tag{33}$$

i. Show that

$$a^\alpha = \gamma\left(c\frac{d\gamma}{dt}, \mathbf{v}\frac{d\gamma}{dt} + \gamma\mathbf{a}\right), \tag{34}$$

where

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{35}$$

is the acceleration that we use in Newtonian physics.

ii. Starting from Eq. (25) and taking derivative with respect to proper time show that

$$u^\alpha a_\alpha = 0. \tag{36}$$

Thus, conclude that four-acceleration is space-like.

iii. Further, using the explicit form of $u^\alpha a_\alpha$ in Eq. (36) derive the identity

$$\frac{d\gamma}{dt} = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\right)\gamma^3. \tag{37}$$

iv. Show that

$$a^\alpha = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4, \mathbf{a}\gamma^2 + \frac{\mathbf{v}}{c}\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4\right) \tag{38}$$

v. Write down the form of the acceleration four-vector in the rest frame ($\mathbf{v} = 0$) of the particle as $(0, \mathbf{a}_0)$, where

$$\mathbf{a}_0 = \mathbf{a}\Big|_{\text{rest frame}} \tag{39}$$

is defined as the proper acceleration. Note that the proper acceleration is a Lorentz invariant quantity, that is, independent of which observer makes the measurement.

vi. Evaluate the following identities involving the proper acceleration

$$a^\alpha a_\alpha = \mathbf{a}_0\cdot\mathbf{a}_0 = \left[\mathbf{a}\cdot\mathbf{a} + \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\right)^2\gamma^2\right]\gamma^4 = \left[\mathbf{a}\cdot\mathbf{a} - \left(\frac{\mathbf{v}\times\mathbf{a}}{c}\right)^2\right]\gamma^6. \tag{40}$$

vii. In a particular frame, if $\mathbf{v}\,||\,\mathbf{a}$ (corresponding to linear motion), deduce

$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^3. \tag{41}$$

And, in a particular frame, if $\mathbf{v}\perp\mathbf{a}$ (corresponding to circular motion), deduce

$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^2. \tag{42}$$

(d) Force: The force four-vector is defined as

$$f^\alpha = c\frac{dp^\alpha}{ds} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \mathbf{F}\gamma\right), \tag{43}$$

where the force $\mathbf{F}$, identified (or defined) as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{44}$$

is the force in Newtonian physics. Starting from Eq. (31) derive the relation

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} \tag{45}$$

which is the power output or the rate of work done by the force $\mathbf{F}$ on the particle.

(e) Equations of motion: The relativistic generalization of Newton's laws are

$$f^\alpha = ma^\alpha. \tag{46}$$

Show that these involve the relations, using the definitions of energy and momentum in Eqs. (28),

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}\gamma + m\mathbf{v}\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\gamma^3, \tag{47a}$$

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} = m\mathbf{v}\cdot\mathbf{a}\gamma^3. \tag{47b}$$

Discuss the non-relativistic limits of the equations of motion.