Homework No. 07 (2024 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2024 Mar 26, 4.30pm

1. (**20 points.**) (Refer Landau and Lifshitz, Problem 1 in Chapter 3.) A simple pendulum consists of a particle of mass m suspended by a massless rod of length l in a uniform gravitational field g.

   (a) Identify the two forces acting on the pendulum to be the force of gravity $m\mathbf{g}$ and the force of tension $\mathbf{T}$. Thus, deduce the Newton equation of motion to be

   $$m\mathbf{a} = m\mathbf{g} + \mathbf{T}, \tag{1}$$

   where $\mathbf{a}$ is acceleration of mass m. Starting from Eq. (1) derive the equation of motion for the simple pendulum

   $$\frac{d^2\phi}{dt^2} = -\omega_0^2 \sin\phi, \tag{2}$$

   where

   $$\omega_0 = \frac{2\pi}{T_0} = \sqrt{\frac{g}{l}}. \tag{3}$$

   (b) Starting from Eq. (2) derive the statement of conservation of energy for this system,

   $$\frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi = \text{constant}. \tag{4}$$

   Hint: Multiply Eq. (2) by $\dot{\phi}$ and express the equation as a total derivative with respect to time.

   (c) For initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$ show that

   $$\frac{1}{2} m l^2 \dot{\phi}^2 - mgl\cos\phi = -mgl\cos\phi_0. \tag{5}$$

   Thus, derive

   $$\frac{dt}{T_0} = \frac{1}{2\pi} \frac{d\phi}{\sqrt{2(\cos\phi - \cos\phi_0)}} \tag{6}$$

   where $T_0 = 2\pi\sqrt{l/g}$.

(d) The time period of oscillations of the simple pendulum is equal to four times the time taken between $\phi = 0$ and $\phi = \phi_0$. Thus, show that

$$T = 4\frac{T_0}{2\pi}\int_0^{\phi_0} \frac{d\phi}{\sqrt{2(\cos\phi - \cos\phi_0)}} \tag{7}$$

$$= \frac{T_0}{\pi}\int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}}}. \tag{8}$$

Then, substitute $\sin\theta = \sin(\phi/2)/\sin(\phi_0/2)$ to determine the period of oscillations of the simple pendulum as a function of the amplitude of oscillations ϕ_0 to be

$$T = T_0\,\frac{2}{\pi}K\left(\sin\frac{\phi_0}{2}\right), \tag{9}$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} \tag{10}$$

is the complete elliptic integral of the first kind.

(e) Using the power series expansion

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} \tag{11}$$

show that for small oscillations ($\phi_0/2 \ll 1$)

$$T = T_0\left[1 + \frac{\phi_0^2}{16} + \ldots\right]. \tag{12}$$

(f) Estimate the percentage error made in the approximation $T \sim T_0$ for $\phi_0 \sim 60°$.

(g) Plot the time period T of Eq. (9) as a function of ϕ_0. What can you conclude about the time period for $\phi_0 = \pi$?

2. (**20 points.**) Consider the differential equation

$$\ddot{x}(t) = -\omega_1^2 x(t), \tag{13}$$

where dot denotes differentiation with respect to time, in conjunction with a suitable initial condition.

(a) Using Fourier transform

$$x(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega t}\tilde{x}(\omega), \tag{14a}$$

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} dt\,e^{i\omega t}x(t), \tag{14b}$$

show that $\tilde{x}(\omega)$ satisfies the algebraic equation

$$-\omega^2\tilde{x}(\omega) = -\omega_1^2\tilde{x}(\omega). \tag{15}$$

Observe that we can arrive at this equation using the transcription,

$$\frac{\partial}{\partial t} \to -i\omega, \tag{16a}$$
$$x(t) \to \tilde{x}(\omega), \tag{16b}$$

in the original differential equation. Thus, the algebraic equation for $\tilde{x}(\omega)$ is

$$(\omega^2 - \omega_1^2)\tilde{x}(\omega) = 0. \tag{17}$$

(b) The solution to the above algebraic equation can be expressed in the form

$$\tilde{x}(\omega) = \tilde{\alpha}(\omega)\delta(\omega^2 - \omega_1^2), \tag{18}$$

where $\tilde{\alpha}(\omega)$ is to be determined. Using the property of δ-functions show that

$$\tilde{x}(\omega) = \frac{\tilde{\alpha}(\omega_1)}{2\omega_1}\delta(\omega - \omega_1) + \frac{\tilde{\alpha}(-\omega_1)}{2\omega_1}\delta(\omega + \omega_1). \tag{19}$$

(c) Using Fourier transform evaluate

$$x(t) = \frac{1}{2\pi}\frac{\tilde{\alpha}(\omega_1)}{2\omega_1}e^{-i\omega_1 t} + \frac{1}{2\pi}\frac{\tilde{\alpha}(-\omega_1)}{2\omega_1}e^{i\omega_1 t}. \tag{20}$$

In terms of numbers

$$A_1 = \frac{1}{2\pi}\frac{\tilde{\alpha}(\omega_1)}{2\omega_1}, \tag{21a}$$
$$B_1 = \frac{1}{2\pi}\frac{\tilde{\alpha}(-\omega_1)}{2\omega_1}, \tag{21b}$$

express the solution in the form

$$x(t) = A_1 e^{-i\omega_1 t} + B_1 e^{i\omega_1 t}. \tag{22}$$

The numbers A_1 and B_1 are determined from initial conditions. For example, show that for initial conditions $x(0) = A$ and $\dot{x}(0) = 0$ the solution is

$$x(t) = A\cos\omega_1 t. \tag{23}$$

3. (**20 points.**) Consider the set of differential equations

$$\ddot{x}_1(t) + \omega_1^2 x_1(t) = \omega_3^2 x_2(t), \tag{24a}$$
$$\ddot{x}_2(t) + \omega_2^2 x_2(t) = \omega_3^2 x_1(t), \tag{24b}$$

where dot denotes differentiation with respect to time, in conjunction with suitable initial conditions.

(a) Using Fourier transform show that $\tilde{x}_1(\omega)$ and $\tilde{x}_2(\omega)$ satisfy the algebraic equations

$$(\omega_1^2 - \omega^2)\tilde{x}_1(\omega) - \omega_3^2\tilde{x}_2(\omega) = 0, \tag{25a}$$
$$-\omega_3^2\tilde{x}_1(\omega) + (\omega_2^2 - \omega^2)\tilde{x}_2(\omega) = 0. \tag{25b}$$

Observe that they decouple for $\omega_3 = 0$. The explicit nature of the coupling is brought out by writing the solutions, $\tilde{x}_1(\omega)$ and $\tilde{x}_2(\omega)$, in the form

$$\tilde{x}_1(\omega) = \frac{\omega_3^2}{(\omega_1^2 - \omega^2)}\tilde{x}_2(\omega), \tag{26a}$$
$$\tilde{x}_2(\omega) = \frac{\omega_3^2}{(\omega_2^2 - \omega^2)}\tilde{x}_1(\omega). \tag{26b}$$

Using the two solutions in conjunction show that the solutions satisfy

$$(\omega - \lambda_1)(\omega + \lambda_1)(\omega - \lambda_2)(\omega + \lambda_2)\tilde{x}_1(\omega) = 0, \tag{27a}$$
$$(\omega - \lambda_1)(\omega + \lambda_1)(\omega - \lambda_2)(\omega + \lambda_2)\tilde{x}_2(\omega) = 0, \tag{27b}$$

where $\pm\lambda_1$ and $\pm\lambda_2$ are roots of the quartic equation

$$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) - \omega_3^4 = 0. \tag{28}$$

Evaluate the roots for $\omega_2^2 > \omega_1^2$ to be

$$\lambda_2^2 = \frac{(\omega_2^2 + \omega_1^2)}{2} + \frac{1}{2}\sqrt{(\omega_2^2 - \omega_1^2)^2 + 4\omega_3^4}, \tag{29a}$$
$$\lambda_1^2 = \frac{(\omega_2^2 + \omega_1^2)}{2} - \frac{1}{2}\sqrt{(\omega_2^2 - \omega_1^2)^2 + 4\omega_3^4}, \tag{29b}$$

and express them in the form

$$\lambda_2^2 = \omega_2^2 + (\mu^2 - \Delta^2), \tag{30a}$$
$$\lambda_1^2 = \omega_1^2 - (\mu^2 - \Delta^2), \tag{30b}$$

where

$$\Delta^2 = \frac{(\omega_2^2 - \omega_1^2)}{2} \tag{31}$$

and

$$\mu^2 = \sqrt{\Delta^4 + \omega_3^4}. \tag{32}$$

Determine the normal modes λ_1 and λ_2 for $\omega_3 = 0$.

(b) Derive the following. The difference in the square of roots,

$$\lambda_2^2 - \lambda_1^2 = 2\mu^2, \tag{33}$$

and the change in the normal modes due to coupling,

$$\omega_1^2 - \lambda_1^2 = (\mu^2 - \Delta^2), \tag{34a}$$
$$\omega_1^2 - \lambda_2^2 = -(\mu^2 + \Delta^2), \tag{34b}$$

and

$$\omega_2^2 - \lambda_1^2 = (\mu^2 + \Delta^2), \tag{35a}$$
$$\omega_2^2 - \lambda_2^2 = -(\mu^2 - \Delta^2). \tag{35b}$$

Using the above relations together with

$$\omega_3^2 = \sqrt{(\mu^2 + \Delta^2)(\mu^2 - \Delta^2)} \tag{36}$$

derive

$$\frac{\omega_3^2}{(\omega_1^2 - \lambda_1^2)} = \sqrt{\frac{\mu^2 + \Delta^2}{\mu^2 - \Delta^2}} = -\frac{\omega_3^2}{(\omega_2^2 - \lambda_2^2)}, \tag{37a}$$
$$\frac{\omega_3^2}{(\omega_2^2 - \lambda_1^2)} = \sqrt{\frac{\mu^2 - \Delta^2}{\mu^2 + \Delta^2}} = -\frac{\omega_3^2}{(\omega_1^2 - \lambda_2^2)}. \tag{37b}$$

(c) Argue that the solutions for the algebraic expressions in Eqs. (27) can be expressed in the form

$$\tilde{x}_1(\omega) = \tilde{a}_1(\omega)\delta\big((\omega - \lambda_1)(\omega + \lambda_1)(\omega - \lambda_2)(\omega + \lambda_2)\big), \tag{38a}$$
$$\tilde{x}_2(\omega) = \tilde{a}_2(\omega)\delta\big((\omega - \lambda_1)(\omega + \lambda_1)(\omega - \lambda_2)(\omega + \lambda_2)\big), \tag{38b}$$

where $\tilde{a}_1(\omega)$ and $\tilde{a}_2(\omega)$ are arbitrary. Using the property of δ-functions show that

$$\frac{\tilde{x}_1(\omega)}{2\pi} = \frac{1}{8\pi\mu^2}\bigg[\frac{\tilde{a}_1(\lambda_1)}{\lambda_1}\delta(\omega - \lambda_1) + \frac{\tilde{a}_1(-\lambda_1)}{\lambda_1}\delta(\omega + \lambda_1) + \frac{\tilde{a}_1(\lambda_2)}{\lambda_2}\delta(\omega - \lambda_2) + \frac{\tilde{a}_1(-\lambda_2)}{\lambda_2}\delta(\omega + \lambda_2)\bigg], \tag{39a}$$
$$\frac{\tilde{x}_2(\omega)}{2\pi} = \frac{1}{8\pi\mu^2}\bigg[\frac{\tilde{a}_2(\lambda_1)}{\lambda_1}\delta(\omega - \lambda_1) + \frac{\tilde{a}_2(-\lambda_1)}{\lambda_1}\delta(\omega + \lambda_1) + \frac{\tilde{a}_2(\lambda_2)}{\lambda_2}\delta(\omega - \lambda_2) + \frac{\tilde{a}_2(-\lambda_2)}{\lambda_2}\delta(\omega + \lambda_2)\bigg]. \tag{39b}$$

The aribitrary coefficients are related due to the coupling in Eqs. (26). Thus, verify that

$$\tilde{a}_1(\pm\lambda_1) = \sqrt{\frac{\mu^2 + \Delta^2}{\mu^2 - \Delta^2}}\tilde{a}_2(\pm\lambda_1), \tag{40a}$$
$$\tilde{a}_1(\pm\lambda_2) = -\sqrt{\frac{\mu^2 - \Delta^2}{\mu^2 + \Delta^2}}\tilde{a}_2(\pm\lambda_2), \tag{40b}$$

and

$$\tilde{a}_2(\pm\lambda_1) = \sqrt{\frac{\mu^2 - \Delta^2}{\mu^2 + \Delta^2}}\tilde{a}_1(\pm\lambda_1), \tag{41a}$$

$$\tilde{a}_2(\pm\lambda_2) = -\sqrt{\frac{\mu^2 + \Delta^2}{\mu^2 - \Delta^2}}\tilde{a}_1(\pm\lambda_2). \tag{41b}$$

Using Eqs. (39) in the Fourier transform

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{x}_1(\omega) \tag{42}$$

and the redefinitions

$$A_1 = \frac{1}{8\pi\mu^2}\frac{\tilde{a}_1(\lambda_1)}{\lambda_1} \qquad B_1 = \frac{1}{8\pi\mu^2}\frac{\tilde{a}_1(-\lambda_1)}{\lambda_1}, \tag{43a}$$

$$A_2 = \frac{1}{8\pi\mu^2}\frac{\tilde{a}_1(\lambda_2)}{\lambda_2} \qquad B_2 = \frac{1}{8\pi\mu^2}\frac{\tilde{a}_1(-\lambda_2)}{\lambda_2}, \tag{43b}$$

which are determined by initial conditions, show that

$$x_1(t) = A_1 e^{-i\lambda_1 t} + B_1 e^{i\lambda_1 t} + A_2 e^{-i\lambda_2 t} + B_2 e^{i\lambda_2 t}, \tag{44a}$$

$$x_2(t) = \sqrt{\frac{\mu^2 - \Delta^2}{\mu^2 + \Delta^2}}\Big[A_1 e^{-i\lambda_1 t} + B_1 e^{i\lambda_1 t}\Big] - \sqrt{\frac{\mu^2 + \Delta^2}{\mu^2 - \Delta^2}}\Big[A_2 e^{-i\lambda_2 t} + B_2 e^{i\lambda_2 t}\Big]. \tag{44b}$$

(d) For initial conditions

$$x_1(0) = A, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 0, \tag{45}$$

show that

$$x_1(t) = \frac{A}{2}\left[\left(1 + \frac{\Delta^2}{\mu^2}\right)\cos\lambda_1 t + \left(1 - \frac{\Delta^2}{\mu^2}\right)\cos\lambda_2 t\right], \tag{46a}$$

$$x_2(t) = \frac{A}{2}\frac{\omega_3^2}{\mu^2}\Big[\cos\lambda_1 t - \cos\lambda_2 t\Big]. \tag{46b}$$

Sympathetic oscillations are characterized by the case

$$\Delta^2 \ll \omega_3^2 \tag{47}$$

when

$$\left(1 \pm \frac{\Delta^2}{\mu^2}\right) \sim 1, \quad \frac{\omega_3^2}{\mu^2} \sim 1, \quad \lambda_2^2 \sim \omega_2^2 + \omega_3^2, \quad \lambda_1^2 \sim \omega_1^2 - \omega_3^2, \tag{48}$$

and

$$x_1(t) = \frac{A}{2}\Big[\cos\lambda_1 t + \cos\lambda_2 t\Big] = A\cos\left(\frac{\lambda_1 - \lambda_2}{2}\right)t\cos\left(\frac{\lambda_1 + \lambda_2}{2}\right)t, \tag{49a}$$

$$x_2(t) = \frac{A}{2}\Big[\cos\lambda_1 t - \cos\lambda_2 t\Big] = A\sin\left(\frac{\lambda_1 - \lambda_2}{2}\right)t\cos\left(\frac{\lambda_1 + \lambda_2}{2}\right)t. \tag{49b}$$

Plot $x_1(t)$ anf $x_2(t)$ for $\omega_2 = 1.01\omega_1$ and $\omega_3 = 0.3\omega_1$, corresponding to $\omega_3 \sim 10\Delta$.