Homework No. 06 (2024 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2024 Mar 19, 4.30pm

1. (**20 points.**) Spherical pendulum: Consider a pendulum that is suspended such that a mass m is able to move freely on the surface of a sphere of radius a (the length of the pendulum). The mass is then subject to the constraint

$$\phi = \frac{1}{2}(\mathbf{r}\cdot\mathbf{r} - a^2) = 0, \tag{1}$$

where a factor of 1/2 is introduced anticipating cancellations. Consider the Lagrangian function

$$L(\mathbf{r},\mathbf{v}) = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + m\mathbf{g}\cdot\mathbf{r} + \mathbf{T}\cdot\boldsymbol{\nabla}\phi. \tag{2}$$

Here ϕ represents the equation for the surface of constraint, such that the gradient $\boldsymbol{\nabla}\phi$ is normal to the surface. The Lagrange multiplier $\mathbf{T}$ is interpreted as the force that is entrusted with the task of keeping the mass on the surface during motion. In this example of spherical pendulum $\mathbf{T}$ is the force of tension. My recording on the topic of planar pendulum, available at

https://youtu.be/dTU9p9VyeqE (45 minute video),

is a resource.

(a) Evaluate the gradient $\boldsymbol{\nabla}$ of the condition of constraint. Show that

$$\boldsymbol{\nabla}\phi = \mathbf{r}. \tag{3}$$

(Hint: Use $\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}$.) Thus, show that

$$\mathbf{T}\cdot\boldsymbol{\nabla}\phi = \mathbf{T}\cdot\mathbf{r} \tag{4}$$

and

$$L(\mathbf{r},\mathbf{v}) = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + m\mathbf{g}\cdot\mathbf{r} + \mathbf{T}\cdot\mathbf{r}. \tag{5}$$

(b) Using the Euler-Lagrange equations derive the equations of motion

$$m\mathbf{a} = m\mathbf{g} + \mathbf{T}, \tag{6}$$

where $\mathbf{a}$ is acceleration of mass m. Comparing Eq. (6) with the Newton equation of motion we recognize the Lagrangian multiplier to be the force of tension. In particular, this specifies the direction of $\mathbf{T}$ to be in the radially inward direction.

i. Equation of constraint: Find the projection of Newton's law of motion along the direction normal to the surface of constraint. Since $\hat{\mathbf{r}}$ is normal to the surface of the sphere we have

$$m\mathbf{a}\cdot\hat{\mathbf{r}} = m\mathbf{g}\cdot\hat{\mathbf{r}} + \mathbf{T}\cdot\hat{\mathbf{r}}, \tag{7}$$

which corresponds to

$$-m\dot{\phi}^2 a = mga\cos\phi + \mathbf{T}\cdot\hat{\mathbf{r}}. \tag{8}$$

ii. Equation of motion: By projecting in the tangential direction $\hat{\boldsymbol{\phi}}$ derive the equation of motion

$$a\ddot{\phi} = -g\sin\phi. \tag{9}$$

(c) Evaluate the canonical momentum

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v}. \tag{10}$$

(d) Construct the Hamiltonian using

$$H(\mathbf{r},\mathbf{p}) = \mathbf{v}\cdot\mathbf{p} - L(\mathbf{r},\mathbf{v}) \tag{11}$$

to be

$$H(\mathbf{r},\mathbf{p}) = \frac{p^2}{2m} - m\mathbf{g}\cdot\mathbf{r} - \mathbf{T}\cdot\mathbf{r}. \tag{12}$$

Derive the Hamilton equations of motion to be

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m}, \tag{13a}$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{r}} = m\mathbf{g} + \mathbf{T}. \tag{13b}$$

Derive the statement of conservation of energy

$$\frac{dH}{dt} = 0 \tag{14}$$

starting from the Hamiltonian in Eq. (12) and using Hamilton equations of motion. You will also need to prove

$$\mathbf{r}\cdot\frac{d\mathbf{T}}{dt} = 0. \tag{15}$$

(e) Show that the angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ satisfies the equation of motion

$$\frac{d\mathbf{L}}{dt} = \mathbf{r}\times m\mathbf{g}, \tag{16}$$

and the angular momentum in the direction of $\mathbf{g}$ is conserved, that is,

$$\frac{d}{dt}(\mathbf{g}\cdot\mathbf{L}) = 0. \tag{17}$$

Area swept out by a particle as it moves along it's trajectory is given by

$$\frac{1}{2}\mathbf{r} \times d\mathbf{r}. \tag{18}$$

The rate at which this area changes is called the areal velocity. Thus, angular momentum is a measure of areal velocity. So, conclude the conservation of areal velocity in the direction of $\mathbf{g}$.

2. (**20 points.**) The Atwood machine consists of two masses m_1 and m_2 connected by a massless (inextensible) string passing over a massless pulley. See Figure 1. Massless pulley implies that tension in the string on both sides of the pulley is the same, say T. Further, the string being inextensible implies that the magnitude of the accelerations of both the masses are the same. Let $m_2 > m_1$.

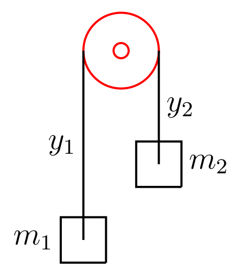

Figure 1: example

(a) Let lengths y_1 and y_2 be positive distances from the pulley to the masses such that the accelerations $a_1 = \ddot{y}_1$ and $a_2 = \ddot{y}_2$ satisfy $a_2 = -a_1 = a$. Using Newton's law determine the equations of motion for the masses to be

$$m_2 g - T = m_2 a, \tag{19a}$$
$$m_1 g - T = -m_1 a. \tag{19b}$$

Thus, show that

$$\text{Equation of motion:} \qquad a = \left(\frac{m_2 - m_1}{m_2 + m_1}\right) g, \tag{20a}$$
$$\text{Equation of constraint:} \qquad T = \frac{2m_1 m_2 g}{(m_1 + m_2)}. \tag{20b}$$

(b) The constraint among the dynamical variables y_1 and y_2 is

$$y_1 + y_2 = L, \tag{21}$$

where L is the total length of the string connecting the two masses. Show that the Lagrangian for Atwood's machine can be expressed in terms of a single dynamical variable, say y_2, as

$$L(y_2, \dot{y}_2) = \frac{1}{2}(m_1 + m_2)\dot{y}_2^2 + (m_2 - m_1)gy_2. \tag{22}$$

Find the corresponding Euler-Lagrange equation.

(c) Using the idea of Lagrange multiplier construct another Lagrangian

$$L(y_1, y_2, \dot{y}_1, \dot{y}_2) = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + m_1 g y_1 + m_2 g y_2 - T\frac{\partial}{\partial y_1}(y_1 + y_2 - L)^2\frac{1}{2}, \tag{23}$$

where T here is interpreted as the Lagrangian multiplier. Find the corresponding Euler-Lagrange equations.

3. (**20 points.**) Consider a wheel rolling on a horizontal surface. See Figure 2. The following

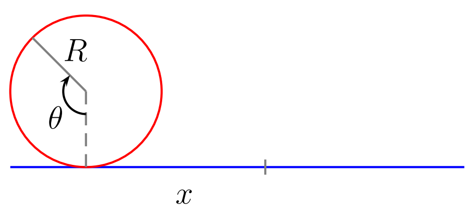

Figure 2: Problem 3.

distinct types of motion are possible for the wheel:

$$\begin{aligned} x &< \theta R, && \text{slipping (e.g. in snow),} \\ x &= \theta R, && \text{perfect rolling,} \\ x &> \theta R, && \text{sliding (e.g. on ice).} \end{aligned} \tag{24}$$

Differentiation of the these relations leads to the characterizations, $v < \omega R$, $v = \omega R$, and $v > \omega R$, respectively, where $v = \dot{x}$ is the linear velocity and $\omega = \dot{\theta}$ is the angular velocity. Assuming the wheel is perfectly rolling, at a given instant of time, the tendency of motion could be to slip, to keep on perfectly rolling, or to slide.

Deduce that while perfectly rolling the relative motion of the point on the wheel that is in contact with the surface with respect to the surface is exactly zero. Thus, conclude that the force of friction on the wheel is zero. The analogy is a mass at rest on a horizontal surface. However, while perfectly rolling, it is possible to have the tendency to slip or slide without actually slipping of sliding. The analogy is that of a mass at rest under the action of an external force and the force of friction. In these cases the force of friction is that of static friction and it acts in the forward or backward direction.

In the following we differentiate between the following:

(a) Tendency of the wheel is to slip (without actually slipping) while perfectly rolling.

(b) Tendency of the wheel is to keep on perfectly rolling.

(c) Tendency of the wheel is to slide (without actually sliding) while perfectly rolling.

Deduce the direction of the force of friction in the above cases. Determine if the friction is working against linear acceleration or angular acceleration.

Perfect rolling involves the contraint $x = \theta R$. Thus, using the D'Alembert's principle and idea of Lagrange multiplier we can write the Lagragian for a perfectly rolling wheel on a horizontal surface to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 - F_s(x - \theta R), \tag{25}$$

where m is the mass of the wheel, I is the moment of inertia of the wheel, and F_s is the Lagrangian multiplier. Using D'Alembert's principle give an interpretation for the Lagrangian multiplier F_s. What is the dimension of F_s? Infer the sign of F_s for the cases when the tendency of the wheel is to slip or slide while perfectly rolling.