Homework No. 05 (2024 Spring)
PHYS 510: CLASSICAL MECHANICS
School of Physics and Applied Physics, Southern Illinois University–Carbondale
Due date: Tuesday, 2024 Mar 5, 4.30pm

1. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a_2. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a frictionless hoop of radius a_1. See Figure 1. For simplification, and at loss of generality, let us chose the motion of the pendulum in the plane containing the hoop.
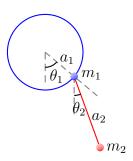

Figure 1: Problem 1.

(a) Determine the Lagrangian for the system to be

$$L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = \frac{1}{2} m_2 a_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 a_2^2 \dot{\theta}_2^2 + m_2 a_1 a_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 g a_1 \cos\theta_1 + m_2 g a_2 \cos\theta_2. \tag{1}$$

(b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_2 a_1^2 \dot{\theta}_1 + m_2 a_1 a_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2), \tag{2a}$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 a_1 a_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g a_1 \sin\theta_1, \tag{2b}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 a_2^2 \dot{\theta}_2 + m_2 a_1 a_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2), \tag{2c}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 a_1 a_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g a_2 \sin\theta_2. \tag{2d}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{\theta}_1 + \omega_1^2 \sin\theta_1 + \frac{1}{\beta}\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{1}{\beta}\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = 0, \tag{3a}$$

$$\ddot{\theta}_2 + \omega_2^2 \sin\theta_2 + \beta\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \beta\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = 0, \tag{3b}$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \tag{4}$$

Note that β is not an independent parameter. Also, observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

2. (**20 points.**) Consider the coplanar double pendulum in Figure 2.
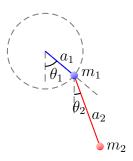

Figure 2: Problem 2.

(a) Write the Lagrangian for the system. In particular, show that the Lagrangian can be expressed in the form

$$L = L_1 + L_2 + L_{\text{int}}, \tag{5}$$

where

$$L_1 = \frac{1}{2}(m_1 + m_2)a_1^2\dot{\theta}_1^2 + (m_1 + m_2)ga_1 \cos\theta_1, \tag{6a}$$

$$L_2 = \frac{1}{2}m_2 a_2^2\dot{\theta}_2^2 + m_2 g a_2 \cos\theta_2, \tag{6b}$$

$$L_{\text{int}} = m_2 a_1 a_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2). \tag{6c}$$

(b) Determine the equations of motion for the system. Express them in the form

$$(m_1 + m_2)a_1\ddot{\theta}_1 + (m_1 + m_2)g \sin\theta_1 + m_2 a_2\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 a_2\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = 0, \tag{7a}$$

$$a_2\ddot{\theta}_2 + g \sin\theta_2 + a_1\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - a_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = 0. \tag{7b}$$