Homework No. 04 (2024 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2024 Feb 20, 4.30pm

1. (**20 points.**) The motion of a particle of mass m undergoing simple harmonic motion is described by

$$\frac{d}{dt}(mv) = -kx, \tag{1}$$

where $v = dx/dt$ is the velocity in the x direction.

(a) Find the Lagrangian for this system that implies the equation of motion of Eq. (1) using Hamilton's principle of stationary action.

(b) Determine the canonical momentum for this system

(c) Determine the Hamiltonian $H(p, x)$ for this system.

(d) Determine the Hamilton equations of motion.

2. (**20 points.**) A relativistic charged particle of charge q and mass m in the presence of a known electric and magnetic field is described by

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{2}$$

(a) Find the Lagrangian for this system, that implies the equation of motion of Eq. (2), to be

$$L(\mathbf{x}, \mathbf{v}, t) = -mc^2\sqrt{1-\frac{v^2}{c^2}} - q\phi + q\mathbf{v}\cdot\mathbf{A}, \tag{3}$$

using Hamilton's principle of stationary action.

(b) Determine the canonical momentum for this system

(c) Determine the Hamiltonian $H(\mathbf{r}, \mathbf{p})$ for this system to be

$$H(\mathbf{x}, \mathbf{p}, t) = \sqrt{m^2c^4 + (\mathbf{p} - q\mathbf{A})^2 c^2} + q\phi. \tag{4}$$

3. (**20 points.**) Consider the action,

(a) in terms of the Lagrangian viewpoint,

$$W[\mathbf{x}] = \int_{t_1}^{t_2} dt \left[\frac{1}{2}m\left(\frac{d\mathbf{x}}{dt}\right)^2 - U(\mathbf{x}, t)\right]. \tag{5}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivative

$$\frac{\delta W}{\delta \mathbf{x}(t)}. \tag{6}$$

Using stationary action principle derive the Euler-Lagrange equations of motion.

(b) in terms of the Hamiltonian viewpoint,

$$W[\mathbf{x},\mathbf{p}] = \int_{t_1}^{t_2} dt \left[\mathbf{p}\cdot\frac{d\mathbf{x}}{dt} - \frac{p^2}{2m} - U(\mathbf{x},t) \right]. \tag{7}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivatives

$$\frac{\delta W}{\delta \mathbf{x}(t)} \quad \text{and} \quad \frac{\delta W}{\delta \mathbf{p}(t)}. \tag{8}$$

Using stationary action principle derive the Hamilton equations of motion.

(c) in terms of the Schwingerian viewpoint,

$$W[\mathbf{x},\mathbf{p},\mathbf{v}] = \int_{t_1}^{t_2} dt \left[\mathbf{p}\cdot\left(\frac{d\mathbf{x}}{dt} - \mathbf{v}\right) + \frac{1}{2}mv^2 - U(\mathbf{x},t) \right]. \tag{9}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivatives

$$\frac{\delta W}{\delta \mathbf{x}(t)}, \quad \frac{\delta W}{\delta \mathbf{v}(t)}, \quad \text{and} \quad \frac{\delta W}{\delta \mathbf{p}(t)}. \tag{10}$$

Using stationary action principle derive the Schwinger equations of motion.

4. (**20 points.**) (Refer Goldstein, 2nd edition, Chapter 1 Problem 14.) As a consequence of the Hamilton's stationary action principle, the equations of motion for a system can be expressed as Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \tag{11}$$

in terms of a Lagrangian $L(x,\dot{x},t)$. Show that the Lagrangian for a system is not unique. In particular, show that if $L(x,\dot{x},t)$ satisfies the Euler-Lagrange equation then

$$L'(x,\dot{x},t) = L(x,\dot{x},t) + \frac{dF(x,t)}{dt}, \tag{12}$$

where $F(x,t)$ is any arbitrary differentiable function, also satisfies the Euler-Lagrange equation.