Homework No. 03 (2024 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2024 Feb 13, 4.30pm

1. (**20 points.**) Fermat's principle in ray optics states that a ray of light takes the path of least time between two given points. Derive Snell's law,

$$n(x)\sin\theta(x) = \eta, \tag{1}$$

where η is a constant, starting from Fermat's principle, for a stratified medium. Here $n(x)$ is the refractive index and $\theta(x)$ is the angle the trajectory of light makes with respect to the x axis.

2. (**20 points.**) Snell's law for refraction for stratified (layered) medium states that

$$n(x)\sin\theta(x) = \eta, \tag{2}$$

where η is a constant. Show that Snell's law can be rewritten in the form

$$\frac{dy}{dx} = \frac{\eta}{\sqrt{n(x)^2 - \eta^2}}. \tag{3}$$

(a) Let us consider a medium with refractive index ($x_1 = a$)

$$n(x) = \begin{cases} 1, & x < a, \\ \dfrac{x}{a}, & a < x. \end{cases} \tag{4}$$

Solve the corresponding differential equation, by substituting $x = \eta\, a \cosh t$, to obtain

$$y(x) - y_0 = \eta\, a \cosh^{-1}\left(\frac{1}{\eta}\frac{x}{a}\right), \qquad a < x. \tag{5}$$

The path in this medium satisfies the equation of a catenary. It is also useful to express the solution in terms of the logarithm as

$$y(x) - y_0 = \eta\, a \ln\left[\frac{1}{\eta}\frac{x}{a} + \sqrt{\left(\frac{1}{\eta}\frac{x}{a}\right)^2 - 1}\,\right], \qquad a < x. \tag{6}$$

(b) For initial conditions, ($x_1 = a$,)

$$y(x_1) = y_1 \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{x=x_1} = y_1' \tag{7}$$

show that integration constants are determined as

$$y_0 = y_1 - \eta\, a \ln\left[\frac{1}{\eta} + \sqrt{\frac{1}{\eta^2} - 1}\right], \qquad \text{and} \qquad \eta = \frac{y_1'}{\sqrt{1 + {y_1'}^2}}. \tag{8}$$

Thus, write the solution as

$$y(x) - y_1 = \eta\, a \ln\left[\frac{\frac{1}{\eta}\frac{x}{a} + \sqrt{\frac{1}{\eta^2}\frac{x^2}{a^2} - 1}}{\frac{1}{\eta} + \sqrt{\frac{1}{\eta^2} - 1}}\right], \qquad a < x. \tag{9}$$

(c) For the special case $y_1 = 0$ and $y_1' \to \infty$ show that $\eta = 1$ and

$$y(x) = a \ln\left[\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right], \qquad a < x. \tag{10}$$

3. (**20 points.**) Find the geodesics on the surface of a circular cylinder. Identify these curves. Hint: To have a visual perception of these geodesics it helps to note that a cylinder can be mapped (or cut open) into a plane.

(a) The distance between two points on the surface of a cylinder of radius a is characterized by the infinitesimal statement

$$ds^2 = a^2 d\phi^2 + dz^2. \tag{11}$$

(b) The geodesic is the extremal of the functional

$$l[z] = \int_{(\phi_1, z_1)}^{(\phi_2, z_2)} ds = \int_{\phi_1}^{\phi_2} a d\phi \sqrt{1 + \left(\frac{1}{a}\frac{dz}{d\phi}\right)^2}. \tag{12}$$

(c) Since the curve passes through the points (z_1, ϕ_1) and (z_2, ϕ_2) we have no variations on the end points. Thus, show that

$$\frac{\delta l[z]}{\delta z(\phi)} = -\frac{d}{d\phi}\left[\frac{\frac{1}{a}\frac{dz}{d\phi}}{\sqrt{1 + \left(\frac{1}{a}\frac{dz}{d\phi}\right)^2}}\right]. \tag{13}$$

(d) Using the extremum principle

$$\frac{\delta l[z]}{\delta z(\phi)} = 0 \tag{14}$$

show that the differential equation for the geodesic is

$$\frac{1}{a}\frac{dz}{d\phi} = c_1, \tag{15}$$

where c_1 is a contant.

(e) Solve the differential equation. Identify the curve described by the solution to be a helix. Illustrate a particular curve using a diagram.

4. (**20 points.**) Consider a rope of uniform mass density $\lambda = dm/ds$ hanging from two points, (x_1, y_1) and (x_2, y_2), as shown in Figure 1. The gravitational potential energy of

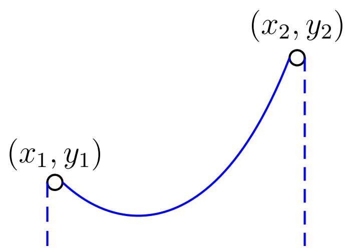

Figure 1: Problem 4.

an infinitely tiny element of this rope at point (x, y) is given by

$$dU = dm\, gy = \lambda g ds\, y, \tag{16}$$

where

$$ds^2 = dx^2 + dy^2. \tag{17}$$

A catenary is the curve that the rope assumes, that minimizes the total potential energy of the rope.

(a) Show that the total potential energy U of the rope hanging between points x_1 and x_2 is given by

$$U[x] = \lambda g \int_{(x_1,y_1)}^{(x_2,y_2)} y ds = \lambda g \int_{y_1}^{y_2} dy\, y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}. \tag{18}$$

(b) Since the curve passes through the points (x_1, y_1) and (x_2, y_2), we have no variations at these (end) points. Thus, show that

$$\frac{\delta U[x]}{\delta x(y)} = -\lambda g \frac{d}{dy}\left[y \frac{\frac{dx}{dy}}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}} \right]. \tag{19}$$

(c) Using the extremum principle show that the differential equation for the catenary is

$$\frac{dx}{dy} = \frac{a}{\sqrt{y^2 - a^2}}, \tag{20}$$

where a is an integration contant.

(d) Show that integration of the differential equation yields the equation of the catenary

$$y = a \cosh \frac{x - x_0}{a}, \tag{21}$$

where x_0 is another integration constant.

(e) For the case $y_1 = y_2$ we have

$$\frac{y_1}{a} = \cosh \frac{x_1 - x_0}{a}, \tag{22a}$$

$$\frac{y_2}{a} = \cosh \frac{x_2 - x_0}{a}, \tag{22b}$$

which leads to, assuming $x_1 \neq x_2$,

$$x_0 = \frac{x_1 + x_2}{2}. \tag{23}$$

Identify x_0 in Figure 1. Next, derive

$$\frac{y_1}{a} = \frac{y_2}{a} = \cosh \frac{x_2 - x_1}{2a}, \tag{24}$$

which, in principle, determines a. However, this is a transcendental equation in a and does not allow exact evaluation of a in closed form and one depends on numerical solutions. Observe that, if $x = x_0$ in Eq. (21), then $y = a$. Identify a in Figure 1.