Midterm Exam No. 01 (2024 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Date: 2024 Feb 22

1. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)} \tag{1}$$

of the following functional,

$$F[u] = \int_a^b dx\, u \sqrt{1 + \left(\frac{du}{dx}\right)^2}, \tag{2}$$

assuming no variation at the end points.

2. (**20 points.**) Using ab initio method, evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)} \tag{3}$$

of the following functional,

$$F[u] = \int_a^b dx \left[1 + b(x)\frac{du(x)}{dx}\right]^{\frac{3}{2}}, \tag{4}$$

assuming no variation at the end points.

3. (**20 points.**) Let the distance between two points in a plane be characterized by the infinitesimal statement

$$ds^4 = dx^4 + dy^4. \tag{5}$$

The geodesic is the extremal of the functional

$$l[y] = \int_{(x_1,y_1)}^{(x_2,y_2)} ds. \tag{6}$$

Find the geodesics. Recognize them.

4. (**20 points.**) The force for linear drag is

$$\mathbf{F} = -b\mathbf{v}. \tag{7}$$

For a mass m falling under uniform gravity in the presence of linear drag, with velocity chosen to be positive for upward direction, we have the equation of motion

$$m\frac{dv}{dt} = -mg - bv. \tag{8}$$

For the case when the mass is moving vertically down, as the mass falls it gains speed and the drag force eventually balances the force of gravity, and from this point on it does not accelerate. Thus, the terminal velocity is defined by requiring $dv/dt = 0$, that is,

$$v_T = \frac{mg}{b} \tag{9}$$

downwards. The equation of motion can be solved for the initial condition of the particle starting from rest, $v(0) = 0$, which leads to the solution

$$v(t) = v_T\Big(e^{-\frac{t}{\tau}} - 1\Big), \tag{10}$$

where

$$\tau = \frac{v_T}{g} \tag{11}$$

sets the scale for time in the problem and is illustrated in Figure 1.

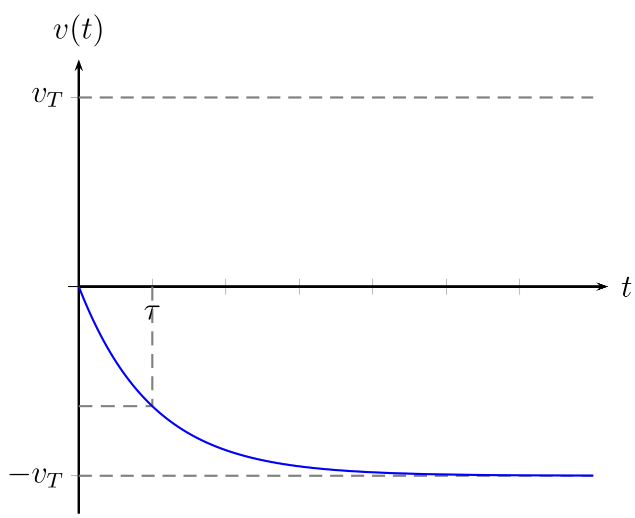

Figure 1: Velocity as a function of time for a mass starting from rest.

(a) Evaluate the solution for the initial condition

$$v(0) = +v_T \tag{12}$$

corresponding to the case when the mass is thrown upwards with terminal velocity. Plot the velocity as a function of time for this case in Figure 1.

(b) Calculate the time when the mass reaches the highest point.

5. (**20 points.**) The geodesics on the surface of a circular cylinder of radius a are helices,

$$z = c_1 a\phi + c_2, \tag{13}$$

where c_1 and c_2 are arbitrary constants and the distance between two points is characterized by the infinitesimal statement

$$ds^2 = a^2 d\phi^2 + dz^2. \tag{14}$$

The geodesic is the extremal of the functional

$$l[z] = \int_{(\phi_1,z_1)}^{(\phi_2,z_2)} ds = \int_{\phi_1}^{\phi_2} a d\phi \sqrt{1 + \left(\frac{1}{a}\frac{dz}{d\phi}\right)^2}. \tag{15}$$

Find the length of a geodesic passing through the points $(\phi_1 = 0, z_1 = 0)$ and $(\phi_2 = \pi, z_2 = \pi a)$.