Homework No. 10 (2022 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: Thursday, 2022 Apr 21, 12.30pm

1. (**20 points.**) The magnetic field associated to radiation fields, in the frequency domain, is given by

$$c\mathbf{B}(\mathbf{r},\omega) = -\hat{\mathbf{r}} \times \mathbf{F}(\theta,\phi;\omega)\,\frac{e^{ikr}}{r}, \tag{1}$$

where

$$\mathbf{F}(\theta,\phi;\omega) = \frac{\mu_0}{4\pi}(-i\omega)\mathbf{J}(\mathbf{k},\omega), \tag{2}$$

where we have used the notation

$$\mathbf{k} = \frac{\omega}{c}\hat{\mathbf{r}}. \tag{3}$$

for insight in the context of Fourier transformation. The associated electric field is given by

$$\mathbf{E}(\mathbf{r},\omega) = -\hat{\mathbf{r}} \times c\mathbf{B}(\mathbf{r},\omega), \tag{4}$$

and satisfies

$$c\mathbf{B}(\mathbf{r},\omega) = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r},\omega). \tag{5}$$

The total energy E radiated into the solid angle $d\Omega$ per unit (positive, $0 \leq \omega < \infty$) frequency range $d\omega$ is given by

$$\frac{\partial}{\partial\omega}\frac{\partial E}{\partial\Omega} = \frac{1}{\pi}\frac{r^2}{c\mu_0}\Big|c\mathbf{B}(\mathbf{r},\omega)\Big|^2. \tag{6}$$

(a) Show that

$$\frac{\partial}{\partial\omega}\frac{\partial E}{\partial\Omega} = \frac{1}{4\pi}\left(\frac{\mu_0 c}{4\pi}\right)\frac{1}{\pi}\Big|\frac{\omega}{c}\hat{\mathbf{r}} \times \mathbf{J}(\mathbf{r},\omega)\Big|^2. \tag{7}$$

Verify that $\omega J/c$ has the dimensions of charge. (Caution: J here is the Fourier transform of current density.) Thus, conclude that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{8}$$

has the dimensions of resistance. Quantum phenomena in electromagnetism is characterized by the Planck's constant h and the associated fine-structure constant

$$\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c}, \tag{9}$$

a dimensionless physical constant. Verify that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \alpha\frac{\hbar}{e^2} = 29.9792458\,\Omega. \tag{10}$$

(b) A loop antenna consists of a circular infinitely thin conductor of radius a carrying a time-dependent current. Let the circular conductor be centered at the origin and placed on the x-y plane such that

$$\mathbf{J}(\mathbf{r}', t') = \hat{\boldsymbol{\phi}}' I_0 \sin\omega_0 t'\, \delta(\rho' - a)\delta(z'), \tag{11}$$

where $\rho' = \sqrt{x'^2 + y'^2}$ and $\hat{\boldsymbol{\phi}}' = -\hat{\mathbf{x}}\sin\phi' + \hat{\mathbf{y}}\cos\phi'$. Evaluate the Fourier transform of the current density using

$$\mathbf{J}(\mathbf{k}, \omega) = \int d^3r' \int dt' e^{-i\mathbf{k}\cdot\mathbf{r}'} e^{i\omega t'} \mathbf{J}(\mathbf{r}', t') \tag{12}$$

and show that

$$\mathbf{J}(\mathbf{k}, \omega) = \hat{\boldsymbol{\phi}}\, 2\pi^2 a I_0\, \delta(\omega - \omega_0)\, J_1(ka\sin\theta), \tag{13}$$

where $J_n(x)$ is the Bessel function of first kind.
Hint: You are expected to encounter the following integral

$$\int_0^{2\pi} d\phi' e^{-ika\sin\theta\cos(\phi-\phi')} \big[-\hat{\mathbf{x}}\sin\phi' + \hat{\mathbf{y}}\cos\phi' \big]. \tag{14}$$

Substitute $\phi' - \phi = \phi''$ to obtain

$$\hat{\boldsymbol{\phi}} \int_0^{2\pi} d\phi'' \cos\phi'' e^{-ika\sin\theta\cos\phi''} - \hat{\boldsymbol{\rho}} \int_0^{2\pi} d\phi'' \sin\phi'' e^{-ika\sin\theta\cos\phi''}. \tag{15}$$

Use the integrals

$$\int_0^{2\pi} \frac{d\phi'}{2\pi} \cos\phi'\, e^{-ix\cos\phi'} = (-i) J_1(x) \tag{16}$$

and

$$\int_0^{2\pi} \frac{d\phi'}{2\pi} \sin\phi'\, e^{-ix\cos\phi'} = 0. \tag{17}$$

We also dropped the delta-function contribution associated to $\delta(\omega + \omega_0)$, because $0 \leq \omega < \infty$.

(c) Show that

$$\frac{\partial}{\partial\omega}\frac{\partial P}{\partial\Omega} = P_0 \pi^2 (ka)^2 J_1^2(ka\sin\theta)\, \delta(\omega - \omega_0), \tag{18}$$

where

$$P_0 = \left(\frac{\mu_0 c}{4\pi}\right) I_0^2. \tag{19}$$

Here we used the interpretation

$$\delta(\omega-\omega_0)\delta(\omega-\omega_0) = \delta(\omega-\omega_0) \int_{-\infty}^{\infty} dt\, e^{i(\omega-\omega_0)t} = \delta(\omega-\omega_0) \int_{-\infty}^{\infty} dt = \delta(\omega-\omega_0) T, \tag{20}$$

where T is the infinite time for which the system is evolving. We used E/T to be the power P.

(d) Integration with respect to frequency yields the power radiated per unit solid angle

$$\frac{\partial P}{\partial \Omega} = P_0 \pi^2 (ka)^2 J_1^2(ka \sin\theta). \tag{21}$$

Plot the angular distribution of radiated power for $ka = 0.5, 2, 3, 4, 6$. Note that

$$ka = \frac{\omega_0}{c} a = 2\pi \frac{a}{\lambda_0}, \tag{22}$$

where λ_0 is the wavelength associated with the angular frequency ω_0.