

Homework No. 09 (2022 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: Thursday, 2022 Apr 14, 12.30pm

1. (20 points.) The magnetic field associated to radiation fields is given by

$$c\mathbf{B}(\mathbf{r}, t) = -\hat{\mathbf{r}} \times \frac{\mu_0 c}{4\pi r} \int d^3 r' \left\{ \frac{1}{c} \frac{\partial}{\partial t'} \mathbf{J}(\mathbf{r}', t') \right\}_{t'=t_r}, \quad (1)$$

where the contribution to the field comes at the retarded time

$$t_r = t - \frac{r}{c} + \hat{\mathbf{r}} \cdot \frac{\mathbf{r}'}{c}. \quad (2)$$

The associated electric field is given by

$$\mathbf{E}(\mathbf{r}, t) = -\hat{\mathbf{r}} \times c\mathbf{B}(\mathbf{r}, t), \quad (3)$$

and satisfies

$$c\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r}, t). \quad (4)$$

From the flux of electromagnetic energy density (the Poynting vector) $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ we deduce the power dP radiated into the solid angle $d\Omega$ to be

$$dP = \lim_{r \rightarrow \infty} r^2 d\Omega \hat{\mathbf{r}} \cdot \mathbf{S}. \quad (5)$$

Using $\hat{\mathbf{r}} \cdot \mathbf{S} = \hat{\mathbf{r}} \cdot (\mathbf{E} \times \mathbf{H}) = (\hat{\mathbf{r}} \times \mathbf{E}) \cdot \mathbf{H}$ show that this leads to the expression

$$\frac{\partial P}{\partial \Omega} = \lim_{r \rightarrow \infty} \frac{1}{4\pi} \left(\frac{\mu_0 c}{4\pi} \right) \left| \frac{c\mathbf{B}(\mathbf{r}, t)}{\frac{\mu_0 c}{4\pi r}} \right|^2 = \lim_{r \rightarrow \infty} \frac{1}{4\pi} \left(\frac{\mu_0 c}{4\pi} \right) |\hat{\mathbf{r}} \times \boldsymbol{\iota}|^2, \quad (6)$$

where we defined the effective current (with direction), using the Greek letter iota,

$$\boldsymbol{\iota} \left(\hat{\mathbf{r}}, t - \frac{r}{c} \right) = \int d^3 r' \left\{ \frac{1}{c} \frac{\partial}{\partial t'} \mathbf{J}(\mathbf{r}', t') \right\}_{t'=t_r}. \quad (7)$$

Verify that $c\mathbf{B}(\mathbf{r}, t) / \left(\frac{\mu_0 c}{4\pi r} \right)$ has the dimensions of current. Thus, conclude that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} = 29.9792458 \Omega \quad (8)$$

has the dimensions of resistance.

- (a) Consider an antenna configuration consisting of parallel current carrying wires of length L , separated by distance a , described in detail by

$$\begin{aligned} \mathbf{J}(\mathbf{r}', t') &= \hat{\mathbf{z}} I_0 \sin \omega_0 t' \delta\left(x' + \frac{a}{2}\right) \delta(y') \theta(-L < 2z' < L). \\ &+ \hat{\mathbf{z}} I_0 \sin \omega_0 t' \delta\left(x' - \frac{a}{2}\right) \delta(y') \theta(-L < 2z' < L). \end{aligned} \quad (9)$$

The function θ equals 1 when the argument is a true statement, and zero otherwise. Show that

$$\begin{aligned} \boldsymbol{\iota}\left(\hat{\mathbf{r}}, t - \frac{r}{c}\right) &= \hat{\mathbf{z}} 2I_0 \cos\left(\omega_0\left(t - \frac{r}{c} - \frac{a}{2c} \sin \theta \cos \phi\right)\right) \frac{\sin\left(\pi \frac{L}{\lambda_0} \cos \theta\right)}{\cos \theta}, \\ &+ \hat{\mathbf{z}} 2I_0 \cos\left(\omega_0\left(t - \frac{r}{c} + \frac{a}{2c} \sin \theta \cos \phi\right)\right) \frac{\sin\left(\pi \frac{L}{\lambda_0} \cos \theta\right)}{\cos \theta}, \end{aligned} \quad (10)$$

where $\omega_0/c = 2\pi/\lambda_0$. Then, evaluate the expression for the magnetic field.

- (b) Using Eq. (6) show that

$$\begin{aligned} \frac{\partial P}{\partial \Omega} &= P_0 \frac{\sin^2 \theta}{\pi} \frac{\sin^2\left(\pi \frac{L}{\lambda_0} \cos \theta\right)}{\cos^2 \theta} \left[\cos\left(\omega_0\left(t - \frac{r}{c} - \frac{a}{2c} \sin \theta \cos \phi\right)\right) \right. \\ &\quad \left. + \cos\left(\omega_0\left(t - \frac{r}{c} + \frac{a}{2c} \sin \theta \cos \phi\right)\right) \right]^2, \end{aligned} \quad (11)$$

where

$$P_0 = \left(\frac{\mu_0 c}{4\pi}\right) I_0^2. \quad (12)$$

Evaluate the average power \bar{P} radiated into a solid angle using

$$\frac{\partial \bar{P}}{\partial \Omega} = \frac{1}{T_0} \int_0^{T_0} dt \frac{\partial P}{\partial \Omega}, \quad (13)$$

where $\omega_0 = 2\pi/T_0$. Show that

$$\frac{\partial \bar{P}}{\partial \Omega} = P_0 \frac{\sin^2 \theta}{\pi} \frac{\sin^2\left(\pi \frac{L}{\lambda_0} \cos \theta\right)}{\cos^2 \theta} 2 \cos^2\left(\pi \frac{a}{\lambda_0} \sin \theta \cos \phi\right). \quad (14)$$

Hint: Use the integral

$$\frac{1}{T_0} \int_0^{T_0} dt \cos^2(\omega_0 t + \delta) = \frac{1}{2}. \quad (15)$$

- (c) For the case $L \ll \lambda_0$, use the approximation

$$\frac{\sin\left(\pi \frac{L}{\lambda_0} \cos \theta\right)}{\cos \theta} \sim \pi \frac{L}{\lambda_0} \quad (16)$$

to obtain

$$\frac{\partial \bar{P}}{\partial \Omega} = P_0 \frac{\sin^2 \theta}{\pi} \left(\pi \frac{L}{\lambda_0}\right)^2 2 \cos^2\left(\pi \frac{a}{\lambda_0} \sin \theta \cos \phi\right). \quad (17)$$

- (d) For the case $\lambda_0 \ll L$, if we restrict our observation region to $\theta \sim \pi/2$, the system has the characteristics of a two dimensional system. To bring this characteristic out we integrate over θ ,

$$\frac{\partial \bar{P}}{\partial \phi} = P_0 \int_0^\pi \sin \theta d\theta \frac{\sin^2 \theta \sin^2 \left(\pi \frac{L}{\lambda_0} \cos \theta \right)}{\pi \cos^2 \theta} 2 \cos^2 \left(\pi \frac{a}{\lambda_0} \sin \theta \cos \phi \right). \quad (18)$$

Substitute

$$z = \pi \frac{L}{\lambda_0} \cos \theta \quad (19)$$

such that

$$-\sin \theta d\theta = \frac{dz}{(\pi L/\lambda_0)} \quad (20)$$

and use the approximations

$$\sin \theta = \sqrt{1 - \frac{z^2}{(\pi L/\lambda_0)^2}} \sim 1 \quad (21)$$

and

$$\pi \frac{L}{\lambda_0} \rightarrow \infty \quad (22)$$

to derive

$$\frac{\partial \bar{P}}{\partial \phi} = P_0 \left(\pi \frac{L}{\lambda_0} \right) 2 \cos^2 \left(\pi \frac{a}{\lambda_0} \cos \phi \right) \frac{1}{\pi} \int_{-\infty}^{\infty} dz \frac{\sin^2 z}{z^2}. \quad (23)$$

Use the integral

$$\int_0^\infty dz \frac{\sin^2 z}{z^2} = \int_0^\infty dz \frac{\sin z}{z} = \frac{\pi}{2}. \quad (24)$$

Thus, derive the expression for the average power radiated per angle $d\phi$,

$$\frac{\partial \bar{P}}{\partial \phi} = P_0 \left(\pi \frac{L}{\lambda_0} \right) 2 \cos^2 \left(\pi \frac{a}{\lambda_0} \cos \phi \right). \quad (25)$$

Compare this with the formula for double-slit interference pattern obtained using the Huygens-Fresnel principle for the classical wave propagation of light.