Homework No. 02 (2022 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: Tuesday, 2022 Feb 1, 4.30pm

1. (**20 points.**) Consider a charge distribution consisting of two point charges with charges equal in magnitude and opposite in sign. The positive charge $+q$ is fixed at position $\mathbf{a} = a\hat{\mathbf{z}}$ on the z axis, and the negative charge $-q$ is fixed at the origin, such that the two charges have a dipole moment $\mathbf{p} = q\mathbf{a}$. The electric field for the configuration is given by

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \frac{(\mathbf{r} - \mathbf{a})}{|\mathbf{r} - \mathbf{a}|^3} - \frac{q}{4\pi\varepsilon_0} \frac{\hat{\mathbf{r}}}{r^2} \tag{1}$$

and the magnetic field $\mathbf{B} = 0$ everywhere.

(a) Starting from the equation for conservation of electromagnetic linear momentum,

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T} + \mathbf{f} = 0, \tag{2}$$

infer that total force along the direction of $\hat{\mathbf{z}}$ on the charges inside volume V is given by

$$\mathbf{F} \cdot \hat{\mathbf{z}} = \int_V d^3r\, \mathbf{f} \cdot \hat{\mathbf{z}} = -\oint_V da\, \hat{\mathbf{r}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}}. \tag{3}$$

(b) Let us choose the volume V to be a sphere of radius r centered at the negative charge. Thus, the points on the surface of this sphere satisfy

$$|\mathbf{r}| = r. \tag{4}$$

Let $r \ll a$. Show that on the surface of volume V

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0} \frac{q}{a^2} \left[\frac{a^2}{r^2} \hat{\mathbf{r}} + \hat{\mathbf{z}} + \mathcal{O}\left(\frac{r}{a}\right) \right]. \tag{5}$$

Using $\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \cos\theta$ show that

$$\hat{\mathbf{r}} \cdot \mathbf{E} = -\frac{1}{4\pi\varepsilon_0} \frac{q}{a^2} \left[\frac{a^2}{r^2} + \cos\theta + \mathcal{O}\left(\frac{r}{a}\right) \right], \tag{6a}$$

$$\mathbf{E} \cdot \hat{\mathbf{z}} = -\frac{1}{4\pi\varepsilon_0} \frac{q}{a^2} \left[\frac{a^2}{r^2} \cos\theta + 1 + \mathcal{O}\left(\frac{r}{a}\right) \right]. \tag{6b}$$

Verify that

$$(\mathbf{E} \cdot \mathbf{D})(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}) = \frac{1}{4\pi} \frac{1}{4\pi\varepsilon_0} \frac{q^2}{a^4} \left[\cos\theta \frac{a^4}{r^4} + 2\cos^2\theta \frac{a^2}{r^2} + \mathcal{O}\left(\frac{a}{r}\right) \right], \tag{7a}$$

$$(\hat{\mathbf{r}} \cdot \mathbf{E})(\mathbf{D} \cdot \hat{\mathbf{z}}) = \frac{1}{4\pi} \frac{1}{4\pi\varepsilon_0} \frac{q^2}{a^4} \left[\cos\theta \frac{a^4}{r^4} + \cos^2\theta \frac{a^2}{r^2} + \frac{a^2}{r^2} + \mathcal{O}\left(\frac{a}{r}\right) \right]. \tag{7b}$$

(c) Show that

$$\mathbf{F}\cdot\hat{\mathbf{z}} = -\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta\, r^2 \left[\frac{1}{2}(\mathbf{E}\cdot\mathbf{D})(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}) - (\hat{\mathbf{r}}\cdot\mathbf{E})(\mathbf{D}\cdot\hat{\mathbf{z}})\right]. \tag{8}$$

We can complete the integral on the azimuth angle to obtain

$$\mathbf{F}\cdot\hat{\mathbf{z}} = \lim_{r\to 0} \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2} \int_0^{\pi} \sin\theta d\theta \frac{2\pi r^2}{4\pi a^2}\left[\cos\theta\,\frac{1}{2}\frac{a^4}{r^4} + \frac{a^2}{r^2} + \mathcal{O}\left(\frac{a}{r}\right)\right]. \tag{9}$$

Thus, show that

$$\mathbf{F}\cdot\hat{\mathbf{z}} = \lim_{r\to 0} \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2} \int_0^{\pi} \sin\theta d\theta \left[\cos\theta\,\frac{1}{4}\frac{a^2}{r^2} + \frac{1}{2}\right]. \tag{10}$$

(d) The first equality in Eq. (10) is divergent in the limit $\delta \to 0$. However it goes to zero if the θ integral is completed before taking the limit. Interpret this argument. The force on the negative charge is

$$\mathbf{F}\cdot\hat{\mathbf{z}} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2}, \tag{11}$$

which is the Coulomb force.

(e) We should also be able to deduce the Coulomb law as the stress on the half-space containing one of the charges. Let us choose the volume V to be the left half space described by $z < a/2$. Then, show that

$$\mathbf{F}\cdot\hat{\mathbf{z}} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \hat{\mathbf{z}}\cdot\mathbf{T}\cdot\hat{\mathbf{z}}\Big|_{z=\frac{a}{2}}. \tag{12}$$

Show that on the surface $z = a/2$ we have

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{aq\,\hat{\mathbf{z}}}{\left(x^2+y^2+\frac{a^2}{4}\right)^{\frac{3}{2}}} \tag{13}$$

and

$$\hat{\mathbf{z}}\cdot\mathbf{T}\cdot\hat{\mathbf{z}}\Big|_{z=\frac{a}{2}} = -\frac{1}{2}\varepsilon_0 E^2. \tag{14}$$

Thus, evaluate the stress on the half space constituting volume V to be

$$\mathbf{F}\cdot\hat{\mathbf{z}} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2}, \tag{15}$$

which is consistent with the interpretation of the Coulomb force.