Final Exam (2022 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

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Date: 2022 Mar 5

1. (**20 points.**) Evalauate the integral

$$\int_{-\infty}^{\infty} dx\, e^{ix}\, \delta(x^2 - a^2) \tag{1}$$

for $a > 0$. Hint: Use the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left| \frac{dF}{dx}\Big|_{x=a_r} \right|}, \tag{2}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$.

2. (**20 points.**) Evaluate the dimension of

$$\frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}. \tag{3}$$

3. (**20 points.**) The free Green dyadic $\boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega)$ satisfies the dyadic differential equation

$$\frac{c^2}{\omega^2}\left[\boldsymbol{\nabla}\boldsymbol{\nabla} - \mathbf{1}\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\right] \cdot \boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = \mathbf{1}\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{4}$$

(a) Show that the divergence of the free Green dyadic is

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = -\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{5}$$

(b) Substitute the divergence in the dyadic differential equation and derive

$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = \left(\boldsymbol{\nabla}\boldsymbol{\nabla} + \frac{\omega^2}{c^2}\mathbf{1}\right)\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{6}$$

(c) Construct the differential equation

$$-(\nabla^2 + k^2)G_0(\mathbf{r}, \mathbf{r}'; \omega) = \delta^{(3)}(\mathbf{r} - \mathbf{r}') \tag{7}$$

for the Green function $G_0(\mathbf{r}, \mathbf{r}'; \omega)$, where

$$k = \frac{\omega}{c}. \tag{8}$$

The free Green function has the (causal) solution

$$G_0(\mathbf{r}-\mathbf{r}';\omega) = \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{9}$$

Show that the free Green dyadic can be expressed in terms of the free Green function as

$$\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = \big[\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}\big]G_0(\mathbf{r},\mathbf{r}';\omega) \tag{10}$$

(d) The free Green dyadic is a function of $\mathbf{r}-\mathbf{r}'$. Thus, we can choose $\mathbf{r}'$ to be the origin without any loss of generality. Substituting $\mathbf{r}\to\mathbf{r}-\mathbf{r}'$ at any moment of the calculation returns the dependence in $\mathbf{r}'$. Evaluate the gradient operators and show that, for $\mathbf{r}'=0$,

$$\boldsymbol{\Gamma}_0(\mathbf{r};\omega) = \frac{e^{ikr}}{4\pi r^3}\Big[-u(ikr)\mathbf{1} + v(ikr)\hat{\mathbf{r}}\hat{\mathbf{r}}\Big], \tag{11}$$

where

$$u(x) = 1 - x + x^2, \tag{12a}$$
$$v(x) = 3 - 3x + x^2. \tag{12b}$$

4. (**20 points.**) The free Green dyadic $\boldsymbol{\Gamma}_0$ can be expressed in terms of the free Green function G_0 as

$$\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = \big[\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}\big]G_0(\mathbf{r},\mathbf{r}';\omega), \tag{13}$$

where

$$G_0(\mathbf{r}-\mathbf{r}';\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{14}$$

In the far-field approximation,

$$r' \ll r, \tag{15}$$

when the observation point $\mathbf{r}$ is very far relative to the source point $\mathbf{r}'$, show that

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'} \sim r - \hat{\mathbf{r}}\cdot\mathbf{r}'. \tag{16}$$

Thus, in the far-field asymptotic limit show that

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \to \frac{e^{ikr}}{4\pi r}\,e^{-i\mathbf{k}'\cdot\mathbf{r}'}, \tag{17}$$

where we introduced the notation

$$\mathbf{k}' = k\,\hat{\mathbf{r}}. \tag{18}$$

Further, the far-field approximation allows the replacement

$$\boldsymbol{\nabla} \to i\mathbf{k}'. \tag{19}$$

Thus, in the far-field approximation show that

$$(\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}) \to (\mathbf{1} - \hat{\mathbf{r}}\hat{\mathbf{r}})k^2 = -\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{1})k^2, \tag{20}$$

which projects vectors in the plane normal to the radial direction. Thus, show that the free Green dyadic in the far-field approximation takes the form

$$\boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = -\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{1}) \frac{k^2}{4\pi} \frac{e^{ikr}}{r} \, e^{-i\mathbf{k}' \cdot \mathbf{r}'}. \tag{21}$$

5. (**20 points.**) The scattering amplitude is given by

$$f(\theta, \phi, \omega) = -\frac{k^2}{4\pi} \chi(\mathbf{k}' - \mathbf{k}, \omega), \tag{22}$$

where $\chi(\mathbf{q}, \omega)$ is the Fourier transform of $\chi(\mathbf{r}, \omega)$,

$$\chi(\mathbf{q}, \omega) = \int d^3r \, e^{i\mathbf{q} \cdot \mathbf{r}} . \chi(\mathbf{r}, \omega) \tag{23}$$

If the obstacles are confined on a plane, say $z = 0$, then it is convenient to define polarizability per unit area $\boldsymbol{\lambda} = \boldsymbol{\alpha}/\text{Area}$,

$$\boldsymbol{\chi}(\mathbf{r}, \omega) = 4\pi \boldsymbol{\lambda}(\mathbf{s}) \, \delta(z), \tag{24}$$

where the δ-function has been used to describe the assumption that the obstacles in a thin film are confined to a plane, $z = 0$ here. Once the obstacles are restricted to be on a plane, we can choose the direction of incidence $\mathbf{k}$ of the plane wave to be normal to the plane. That is, $\mathbf{k} \cdot \mathbf{s} = 0$, where $\mathbf{s}$ are the positions of the point obstacles on the plane. Further, notice that in this special case the electric field $\mathbf{E}_0$ is independent of the position $\mathbf{s}$. Using these considerations show that the scattering amplitude, for isotropic polarizabilities, is given by

$$f(\theta, \phi, \omega) = -k^2 \int d^2s \, e^{ik\hat{\mathbf{r}} \cdot \mathbf{s}} \lambda(\mathbf{s}). \tag{25}$$

For a disc of radius R centered at position $\mathbf{s}_0$ with uniform polarizability per unit area λ complete the integrals to obtain

$$f(\theta, \phi, \omega) = -\lambda k^2 \pi R^2 2 \frac{J_1(kR \sin\theta)}{kR \sin\theta} \, e^{ik\hat{\mathbf{r}} \cdot \mathbf{s}_0}. \tag{26}$$

Hint: Use the integral representation of zeroth order Bessel function of the first kind

$$J_0(t) = \int_0^{2\pi} \frac{d\phi}{2\pi} \, e^{it\cos\phi} \tag{27}$$

and the identity

$$\int_0^b t dt J_0(t) = bJ_1(b). \tag{28}$$

Note the limiting value

$$\lim_{x \to 0} \frac{J_1(x)}{x} = \frac{1}{2}, \tag{29}$$

which guarantees a well defined value for the scattering amplitude at $\theta = 0$. We observe the interesting feature that the scattering amplitude at $\theta = 0$ is entirely given by the area of the disc.