

# Homework No. 10 (2022 Spring)

## PHYS 510: CLASSICAL MECHANICS

*Department of Physics, Southern Illinois University–Carbondale*

Due date: Tuesday, 2022 Apr 19, 4.30pm

1. (20 points.) Resource: Lecture dated 2021 April 13, available at

<https://youtu.be/VGfxAMMkvM4>.

Kepler problem is described by the potential energy

$$U(r) = -\frac{\alpha}{r}, \quad (1)$$

and the corresponding Lagrangian

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}. \quad (2)$$

For the case when the total energy  $E$  is negative,

$$-\frac{\alpha}{2r_0} < E < 0, \quad r_0 = \frac{L_z^2}{\mu\alpha}, \quad (3)$$

where  $L_z$  is the angular momentum, the motion is described by an ellipse,

$$r(\phi) = \frac{r_0}{1 + e \cos(\phi - \phi_0)}, \quad e = \sqrt{1 + \frac{E}{(\alpha/2r_0)}}. \quad (4)$$

Perihelion is the point in the orbit of a planet when it is closest to the Sun. This corresponds to  $\phi = \phi_0$ . The precession of the perihelion is suitably defined in terms of the angular displacement  $\Delta\phi$  of the perihelion during one revolution,

$$\Delta\phi = 2 \left[ \int_{r_{\min}}^{r_{\max}} d\phi \right] - 2\pi, \quad (5)$$

where

$$r_{\min} = \frac{r_0}{1 + e} \quad (6)$$

is the perihelion, when the planet is closest to Sun, and

$$r_{\max} = \frac{r_0}{1 - e} \quad (7)$$

is the aphelion, corresponding to  $\phi = \phi_0 + \pi$ , when the planet is farthest from Sun.

(a) For the Kepler problem derive the relation

$$d\phi = \frac{r_0 dr}{r^2} \frac{1}{\sqrt{e^2 - \left(1 - \frac{r_0}{r}\right)^2}}. \quad (8)$$

Show that the precession of perihelion is zero for the Kepler problem.

(b) When a small correction

$$\delta U(r) = -\frac{\beta}{r^3} = \kappa U_0 \left(\frac{r_0}{r}\right)^3, \quad (9)$$

expressed in terms of dimensionless parameter  $\kappa$  using the relation  $\beta = -\kappa U_0 r_0^3$ , is added we have the perturbed potential energy

$$U(r) = -\frac{\alpha}{r} - \frac{\beta}{r^3} = -\frac{\alpha}{2r_0} \left[ \frac{r_0}{r} + \kappa \left(\frac{r_0}{r}\right)^3 \right]. \quad (10)$$

Show that the precession of the perihelion is no longer zero.