Homework No. 09 (2022 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2022 Apr 12, 4.30pm

1. (**20 points.**) (Based on Schwinger, chapter 9) The Hamiltonian for a Kepler problem is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{\alpha}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{1}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the two constituent particles of masses m_1 and m_2.

(a) Introduce the coordinates representing the center of mass, relative position, total momentum, and relative momentum:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2}, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2}, \tag{2}$$

respectively, to rewrite the Hamiltonian as

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} - \frac{\alpha}{r}, \tag{3}$$

where

$$M = m_1 + m_2, \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{4}$$

(b) Show that Hamilton's equations of motion are given by

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{P}}{M}, \quad \frac{d\mathbf{P}}{dt} = 0, \quad \frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\mu}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\alpha\mathbf{r}}{r^3}. \tag{5}$$

(c) Verify that the Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times \mathbf{L}}{\mu\alpha}, \tag{6}$$

are the three constants of motion for the Kepler problem. That is, show that

$$\frac{dH}{dt} = 0, \quad \frac{d\mathbf{L}}{dt} = 0, \quad \frac{d\mathbf{A}}{dt} = 0. \tag{7}$$