

Homework No. 07 (2022 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2022 Mar 24, 4.30pm

1. (20 points.) Consider infinitesimal rigid translation in space, described by

$$\delta \mathbf{r} = \delta \boldsymbol{\epsilon}, \quad \delta \mathbf{p} = 0, \quad \delta t = 0, \quad (1)$$

where $\delta \boldsymbol{\epsilon}$ is independent of position and time.

- (a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta \boldsymbol{\epsilon}} = - \int_{t_1}^{t_2} dt \frac{\partial H}{\partial \mathbf{r}}. \quad (2)$$

- (b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta \boldsymbol{\epsilon}} = \int_{t_1}^{t_2} dt \frac{d\mathbf{p}}{dt} = \mathbf{p}(t_2) - \mathbf{p}(t_1). \quad (3)$$

- (c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$- \frac{\partial H}{\partial \mathbf{r}} = 0. \quad (4)$$

That is, when the Hamiltonian is independent of position. Or, when the force $\mathbf{F} = -\partial H/\partial \mathbf{r} = 0$.

- (d) Deduce that the linear momentum is conserved, that is,

$$\mathbf{p}(t_1) = \mathbf{p}(t_2), \quad (5)$$

when the action has translation symmetry.

2. (20 points.) Consider infinitesimal rigid translation in time, described by

$$\delta \mathbf{r} = 0, \quad \delta \mathbf{p} = 0, \quad \delta t = \delta \epsilon, \quad (6)$$

where $\delta \epsilon$ is independent of position and time.

- (a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta \epsilon} = - \int_{t_1}^{t_2} dt \frac{\partial H}{\partial t}. \quad (7)$$

- (b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta \epsilon} = - \int_{t_1}^{t_2} dt \frac{dH}{dt} = -H(t_2) + H(t_1). \quad (8)$$

- (c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$- \frac{\partial H}{\partial t} = 0. \quad (9)$$

That is, when the Hamiltonian is independent of time.

- (d) Deduce that the Hamiltonian is conserved, that is,

$$H(t_1) = H(t_2), \quad (10)$$

when the action has translation symmetry.

3. (20 points.) Consider infinitesimal rigid rotation, described by

$$\delta \mathbf{r} = \delta \boldsymbol{\omega} \times \mathbf{r}, \quad \delta \mathbf{p} = \delta \boldsymbol{\omega} \times \mathbf{p}, \quad \delta t = 0, \quad (11)$$

where $d\delta \boldsymbol{\omega}/dt = 0$.

- (a) Show that the variation in the action under the above rotation is

$$\frac{\delta W}{\delta \boldsymbol{\omega}} = \int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial L}{\partial \mathbf{r}} + \mathbf{p} \times \frac{\partial L}{\partial \mathbf{p}} \right] \quad (12)$$

or

$$\frac{\delta W}{\delta \boldsymbol{\omega}} = - \int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial H}{\partial \mathbf{r}} + \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}} \right]. \quad (13)$$

- (b) Show, separately, that the change in the action under the above rotation is also given by

$$\frac{\delta W}{\delta \boldsymbol{\omega}} = \int_{t_1}^{t_2} dt \frac{d\mathbf{L}}{dt} = \mathbf{L}(t_2) - \mathbf{L}(t_1), \quad (14)$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum.

- (c) The system is defined to have rotational symmetry when the action does not change under rigid rotation. Show that a system has rotation symmetry when

$$\mathbf{r} \times \frac{\partial L}{\partial \mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial L}{\partial \mathbf{p}} = 0, \quad (15)$$

or

$$\mathbf{r} \times \frac{\partial H}{\partial \mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}} = 0. \quad (16)$$

Show that this corresponds to

$$\frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \phi} = 0, \quad (17)$$

or

$$\frac{\partial H}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial H}{\partial \phi} = 0. \quad (18)$$

That is, when the Lagrangian is independent of angular coordinates θ and ϕ .

(d) Deduce that the angular momentum is conserved, that is,

$$\mathbf{L}(t_1) = \mathbf{L}(t_2), \quad (19)$$

when the action has rotational symmetry.