Homework No. 06 (2022 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2022 Mar 15, 4.30pm

1. (**20 points.**) The topic of Lagrange multiplier will not be covered in class during this semester. Write a brief summary on Lagrange multiplier. My recording on the topic available at

   https://youtu.be/dTU9p9VyeqE (45 minute video)

   is a resource. The following problem is based on Lagrange multiplier.

2. (**20 points.**) Spherical pendulum: Consider a pendulum that is suspended such that a mass m is able to move freely on the surface of a sphere of radius a (the length of the pendulum). The mass is then subject to the condition of constraint

$$F = \frac{1}{2}(x^2 + y^2 + z^2 - a^2) = 0, \tag{1}$$

   where the factor of 1/2 is introduced anticipating cancellations. Consider the Lagrangian function

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 - mgz - \lambda F. \tag{2}$$

   (a) Evaluate the gradient $\boldsymbol{\nabla}$ of the condition of constraint. Show that

$$\boldsymbol{\nabla} F = \mathbf{r}. \tag{3}$$

   (b) Using the Euler-Lagrange equations derive the equations of motion

$$m\ddot{\mathbf{r}} = -mg\hat{\mathbf{z}} - \lambda\mathbf{r}. \tag{4}$$

   (c) Derive an expression for λ. In particular, show that it can be expressed in the form

$$-\lambda a = \hat{\mathbf{r}} \cdot \mathbf{N}. \tag{5}$$

   Find $\mathbf{N}$. Give the physical interpretation of $\mathbf{N}$ using D'Alembert's principle.

   (d) Show that the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p} = m\dot{\mathbf{r}}$ is the momentum of the particle, about the z-axis is conserved. That is,

$$\frac{d}{dt}(\hat{\mathbf{z}} \cdot \mathbf{L}) = 0. \tag{6}$$

   Show that this also implies the conservation of the areal velocity

$$\frac{dS}{dt} = \frac{1}{2}(x\dot{y} - y\dot{x}), \tag{7}$$

   where S is the area swept out.

(e) Show that

$$\frac{dF}{dt} = \mathbf{r} \cdot \dot{\mathbf{r}} = 0. \tag{8}$$

Using this derive the statement of conservation of energy,

$$\frac{dH}{dt} = 0, \qquad H = \frac{1}{2} m \dot{\mathbf{r}}^2 + mgz, \tag{9}$$

starting from the equation of motion in Eq. (4) and multiplying by $\dot{\mathbf{r}}$.