Homework No. 02 (2022 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2022 Feb 1, 4.30pm

1. (**20 points.**) Given
$$F[u] = \int_{x_1}^{x_2} dx\, a(x) \frac{du(x)}{dx}. \tag{1}$$
Evaluate
$$\frac{\delta F}{\delta u(x)}. \tag{2}$$

2. (**20 points.**) (Gelfand and Fomin, Calculus of Variations.) Evaluate the functional derivatives of the following functionals, assuming no variation at the end points.

   (a)
$$F[y] = \int_0^1 dx\, \frac{dy}{dx} \tag{3}$$

   (b)
$$F[y] = \int_0^1 dx\, y \frac{dy}{dx} \tag{4}$$

   (c)
$$F[y] = \int_0^1 dx\, xy \frac{dy}{dx} \tag{5}$$

   (d)
$$F[y] = \int_a^b \frac{dx}{x^3} \left(\frac{dy}{dx}\right)^2 \tag{6}$$

3. (**20 points.**) Evaluate the functional derivative
$$\frac{\delta F[u]}{\delta u(x)} \tag{7}$$
of the following functionals, assuming no variation at the end points. Given $a(x)$ is a known function.

   (a)
$$F[u] = \int_{x_1}^{x_2} dx\, a(x) \left[1 + \frac{du(x)}{dx} + \frac{d^2u(x)}{dx^2} + \frac{d^3u(x)}{dx^3}\right] \tag{8}$$

(b)

$$F[u] = \int_a^b dx \, \frac{1}{\left(1 + \dfrac{d^3u}{dx^3}\right)} \tag{9}$$

(c)

$$F[u] = \int_a^b dx \, x^5 \sqrt{1 + \frac{d^3u}{dx^3}} \tag{10}$$

(d)

$$F[u] = \int_a^b dx \sqrt{1 + \frac{du}{dx} + \frac{d^3u}{dx^3}} \tag{11}$$

4. (**20 points.**) The distance between two points in a plane is characterized by the infinitesimal statement

$$ds^2 = dx^2 + dy^2. \tag{12}$$

Let us prove the intuitively obvious statement that the curve of shortest distance going through two points (x_1, y_1) and (x_2, y_2) in a plane, the geodesics of a plane, is a straight line passing through the two points. This amounts to finding the extremal of the functional

$$l[y] = \int_{(x_1,y_1)}^{(x_2,y_2)} ds = \int_{(x_1,y_1)}^{(x_2,y_2)} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{13}$$

Using the fact that we require the curve to necessarily pass through the points (x_1, y_1) and (x_2, y_2), show that

$$\frac{\delta l[y]}{\delta y(x)} = -\frac{d}{dx}\left[\frac{\dfrac{dy}{dx}}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}\right]. \tag{14}$$

Then, using the condition that the functional derivative is zero for the extremal curve, derive the equation of the geodesic to be given by

$$\frac{dy}{dx} = c, \tag{15}$$

where c is a contant. Identify this as the equation of a straight line in a plane. Find c.