Midterm Exam No. 02 (2022 Spring)

PHYS 510: CLASSICAL MECHANICS

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1. (**20 points.**) A system, characterized by the parameters ω, α, and β, and the dynamical parameter θ, is described by the equation of motion

$$\ddot{\theta} + \omega^2 \sin\theta + \alpha\ddot{\theta}\sin\theta + \beta\dot{\theta}\sin\theta\cos\theta = 0. \tag{1}$$

Write the above equation of motion in the small angle approximation, to the leading order in θ.

2. (**20 points.**) Given a Lagrangian L, the Hamiltonian H is given by

$$H = \mathbf{p}\cdot\frac{d\mathbf{r}}{dt} - L, \tag{2}$$

where $\mathbf{p}$ is the canonical momentum. Evaluate

$$\frac{\partial H}{\partial \mathbf{v}}, \tag{3}$$

where $\mathbf{v}$ stands for $d\mathbf{r}/dt$.

3. (**30 points.**) (Refer Goldstein, Sec. 9.5.) Hamiltonian for the motion of a ball (along the radial direction) near the surface of Earth is given by

$$H(z, p_z) = \frac{p_z^2}{2m} - mgz. \tag{4}$$

(a) Determine the equations of motions using

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_z} \qquad \text{and} \qquad \frac{dp_z}{dt} = -\frac{\partial H}{\partial z}. \tag{5}$$

Then, solve the coupled differential equations to find the familiar elementary solution

$$z = z_0 + \frac{p_0}{m}t + \frac{1}{2}gt^2. \tag{6}$$

(b) Next, determine the equations of motion using

$$\big[z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{\partial H}{\partial p_z} \qquad \text{and} \qquad \big[p_z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = -\frac{\partial H}{\partial z}. \tag{7}$$

Then, using

$$z = z_0 + t\big[z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0} + \frac{1}{2}t^2\big[\big[z, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0} + \cdots \tag{8}$$

rederive the elementary solution. Here 0 in the subscripts refers to the initial conditions at $t = 0$.

4. (**30 points.**) Hamiltonian for a charge particle of mass m and charge q in a magnetic field $\mathbf{B}$ is given by

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2, \tag{9}$$

where

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{10}$$

Let

$$\frac{\partial \mathbf{A}}{\partial t} = 0. \tag{11}$$

(a) Show that the Hamilton equations of motion leads to the equations, using ($\mathbf{v} = d\mathbf{x}/dt$)

$$m\mathbf{v} = \mathbf{p} - q\mathbf{A}, \tag{12a}$$

$$\frac{d\mathbf{p}}{dt} = q(\boldsymbol{\nabla}\mathbf{A}) \cdot \mathbf{v}. \tag{12b}$$

Show that the above equations in conjunction imply the familiar equation

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}. \tag{13}$$

(b) Evaluate the Poisson braket

$$\big[\mathbf{x}, \mathbf{x}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{14}$$

(c) Evaluate the Poisson braket

$$\big[\mathbf{x}^i, \mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{1}{m}\mathbf{1}^{ij}. \tag{15}$$

(d) Evaluate the Poisson braket

$$\big[\mathbf{x}^i, \mathbf{p}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \mathbf{1}^{ij}. \tag{16}$$

(e) Evaluate the Poisson braket

$$\big[m\mathbf{v}^i, m\mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = q(\boldsymbol{\nabla}^i\mathbf{A}^j - \boldsymbol{\nabla}^j\mathbf{A}^i). \tag{17}$$

Verify that

$$(\boldsymbol{\nabla}^i\mathbf{A}^j - \boldsymbol{\nabla}^j\mathbf{A}^i) = \varepsilon^{ijk}\mathbf{B}^k = -\mathbf{1} \times \mathbf{B}. \tag{18}$$

Poisson bracket in classical mechanics has direct correspondence to commutation relation in quantum mechanics through the factor $i\hbar$, which conforms with experiments and balances the dimensions. Then, we can write

$$\big[m\mathbf{v}, m\mathbf{v}\big] = i\hbar q\mathbf{B} \tag{19}$$

or

$$m\mathbf{v} \times m\mathbf{v} = i\hbar q\mathbf{B}, \tag{20}$$

using the fact that the commutator and the vector product satisfies the same Lie algebra as that of Poisson bracket.

(f) Evaluate the Poisson braket

$$\big[\mathbf{p}, \mathbf{p}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{21}$$