(Preview of) Final Exam (2022 Spring)

PHYS 510: CLASSICAL MECHANICS

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1. (**20 points.**) Not available in preview mode.

2. (**20 points.**) Not available in preview mode.

3. (**20 points.**) The relativity principle states that the laws of physics are invariant (or covariant) when observed using different coordinate systems. In special relativity we restrict these coordinate systems to be uniformly moving with respect to each other.

   (a) Linear: Spatial homogeneity, spatial isotropy, and temporal homogeneity, require the transformation to be linear. (We will skip this derivation. No submission needed.) Then, for simplicity, restricting to coordinate systems moving with respect to each other in a single direction, we can write

$$z' = A(v)\,z + B(v)\,t, \tag{1a}$$
$$t' = E(v)\,z + F(v)\,t. \tag{1b}$$

   We will refer to the respective frames as primed and unprimed.

   (b) Identity: An object at rest in the unprimed frame, described by $z' = 0$, will be described in the primed frame as $z = vt$. Using these in Eq. (1a), we have

$$0 = A(v)\,vt + B(v)\,t. \tag{2}$$

   This implies $B(v) = -vA(v)$. Thus, show that

$$z' = A(v)\,(z - vt), \tag{3a}$$
$$t' = E(v)\,z + F(v)\,t. \tag{3b}$$

   (c) Reversal: The descriptions of a process in the unprimed frame moving to the right with velocity v with respect to the primed should be identical to those made in the unprimed (with their axis flipped) moving with velocity $-v$ with respect to the primed (with their axis flipped). This is equivalent to the requirement of isotropy in an one dimensional space. That is, the transformation must be invariant under

$$z \to -z, \qquad z' \to -z', \qquad v \to -v. \tag{4}$$

This imples

$$-z' = A(-v)\,(-z + vt), \tag{5a}$$
$$t' = -E(-v)\,z + F(-v)\,t. \tag{5b}$$

Show that Eqs. (3) and (5a) in conjunction imply

$$A(-v) = A(v). \tag{6}$$

Further, show that Eqs. (3b) and (5b) in conjunction implies

$$E(-v) = -E(v), \tag{7a}$$
$$F(-v) = F(v). \tag{7b}$$

(d) Reciprocity: The description of a process in the unprimed frame moving to the right with velocity v is identical to the description in the primed frame moving to the left. That is, the transformation must be invariant under

$$(z,t) \to (z',t') \qquad (z',t') \to (z,t) \qquad v \to -v. \tag{8}$$

Show that this implies

$$z = A(-v)\,(z' + vt'), \tag{9a}$$
$$t = E(-v)\,z' + F(-v)\,t'. \tag{9b}$$

Show that Eqs. (3) and Eqs. (9) imply

$$E(v) = \frac{1}{v}\left[\frac{1}{A(v)} - A(v)\right], \tag{10a}$$
$$F(v) = A(v). \tag{10b}$$

Together, for arbitrary $A(v)$, the relativity principle allows the following transformations,

$$z' = A(v)\,(z - vt), \tag{11a}$$
$$t' = A(v)\left[\frac{1}{v}\left(\frac{1}{A(v)^2} - 1\right) z + t\right]. \tag{11b}$$

In Galilean relativity we require $t' = t$. Show that this is obtained with

$$A(v) = 1 \tag{12}$$

in Eqs. (11). This leads to the Galilean transformation

$$z' = z - vt, \tag{13a}$$
$$t' = t. \tag{13b}$$

In Einstein's special relativity the requirement is for a special speed c that is described identically by both the primed and unprimed frames. That is,

$$z = ct, \tag{14a}$$
$$z' = ct'. \tag{14b}$$

Show that Eqs. (14) when substituted in in Eqs. (11) leads to

$$A(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{15}$$

This corresponds to the Lorentz transformation

$$z' = A(v)(z - vt), \tag{16a}$$
$$t' = A(v)\left(-\frac{v}{c^2}z + t\right). \tag{16b}$$

This suggests that it should be possible to contrive additional solutions for $A(v)$ that respects the relativity principle, but with new physical requirements for the respective choice of $A(v)$. Construct one such transformation, which will not be used in grading.

4. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{17a}$$
$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{17b}$$

where

$$E = mc^2\gamma, \tag{18a}$$
$$\mathbf{p} = m\mathbf{v}\gamma, \tag{18b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{19}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{20}$$

and initial conditions

$$\mathbf{v}(0) = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{21a}$$
$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}. \tag{21b}$$

(a) In terms of the definition

$$\boldsymbol{\omega}_0 = \frac{1}{c}\frac{q\mathbf{E}}{m}, \tag{22}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0 \cdot \boldsymbol{\beta} \tag{23}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{24}$$

(b) Since the particle starts from rest show that we have

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t. \tag{25}$$

For our configuration this implies

$$\beta_x = 0, \tag{26a}$$

$$\beta_y\gamma = \omega_0 t, \tag{26b}$$

$$\beta_z = 0. \tag{26c}$$

Further, deduce

$$\beta_y = \frac{\omega_0 t}{\sqrt{1+\omega_0^2 t^2}}. \tag{27}$$

Integrate again and use the initial condition to show that the motion is described by

$$y - y_0 = \frac{c}{\omega_0}\left[\sqrt{1+\omega_0^2 t^2} - 1\right]. \tag{28}$$

Rewrite the solution in the form

$$\left(y - y_0 + \frac{c}{\omega_0}\right)^2 - c^2t^2 = \frac{c^2}{\omega_0^2}. \tag{29}$$

This represents a hyperbola passing through $y = y_0$ at $t = 0$. If we choose the initial position $y_0 = c/\omega_0$ we have

$$y^2 - c^2t^2 = y_0^2. \tag{30}$$

(c) The (constant) proper acceleration associated with this motion is

$$\alpha = \omega_0 c. \tag{31}$$

A Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$y - y_0 = \frac{1}{2}\alpha t^2. \tag{32}$$

Show that the hyperbolic curve

$$y = y_0 \sqrt{1 + \frac{c^2 t^2}{y_0^2}} \tag{33}$$

in regions that satisfy

$$\omega_0 t \ll 1 \tag{34}$$

is approximately the parabolic curve

$$y = y_0 + \frac{1}{2}\alpha t^2 + \dots . \tag{35}$$

5. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{36a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{36b}$$

where

$$E = mc^2\gamma, \tag{37a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{37b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{38}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{39}$$

and initial conditions

$$\mathbf{v}(0) = v_0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{40a}$$

$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}. \tag{40b}$$

We will use the associated definitions $\boldsymbol{\beta}_0 = \mathbf{v}(0)/c$ and $\boldsymbol{\gamma}_0 = 1/\sqrt{1 - \beta_0^2}$.

(a) In terms of the definition

$$\boldsymbol{\omega}_0 = \frac{1}{c}\frac{q\mathbf{E}}{m}, \tag{41}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0 \cdot \boldsymbol{\beta} \tag{42}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{43}$$

(b) For our configuration show that

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t + \beta_0\gamma_0\hat{\mathbf{x}}, \tag{44}$$

such that

$$\beta_x\gamma = \beta_0\gamma_0, \tag{45a}$$
$$\beta_y\gamma = \omega_0 t, \tag{45b}$$
$$\beta_z\gamma = 0. \tag{45c}$$

Using $\beta_z\gamma = 0$, learn that

$$\frac{\beta_z^2}{1-\beta_x^2-\beta_y^2-\beta_z^2} = 0 \tag{46}$$

and in conjunction with $\beta_x\gamma = \beta_0\gamma_0$ deduce that

$$\beta_z = 0 \tag{47}$$

and

$$\frac{\beta_x^2}{\beta_0^2} + \beta_y^2 = 1. \tag{48}$$

Thus, deduce

$$\gamma^2 = \omega_0^2 t^2 + \gamma_0^2 \tag{49}$$

and

$$\beta_x^2 + \beta_y^2 = \beta_0^2 + \frac{\beta_y^2}{\gamma_0^2}. \tag{50}$$

Further, deduce

$$\beta_y = \frac{\bar{\omega}_0 t}{\sqrt{1+\bar{\omega}_0^2 t^2}} \tag{51}$$

and

$$\beta_x = \frac{\beta_0}{\sqrt{1+\bar{\omega}_0^2 t^2}}, \tag{52}$$

where

$$\bar{\omega}_0 = \frac{\omega_0}{\gamma_0}. \tag{53}$$

Integrate again and use the initial condition to show that the motion is described by

$$y - y_0 = \frac{c}{\bar{\omega}_0}\left[\sqrt{1+\bar{\omega}_0^2 t^2} - 1\right], \tag{54a}$$
$$x - x_0 = \frac{v_0}{\bar{\omega}_0}\sinh^{-1}\bar{\omega}_0 t, \tag{54b}$$

and $z = 0$.

(c) Show that for $v_0 = 0$ we reproduce the solution for a particle starting from rest. Next, for

$$\bar{\omega}_0 t \ll 1 \tag{55}$$

and

$$\alpha = \bar{\omega}_0 c \tag{56}$$

obtain the non-relativistic limits,

$$y - y_0 = \frac{1}{2}\alpha t^2, \tag{57a}$$

$$x - x_0 = v_0 t. \tag{57b}$$

Hint: Recall the series expansion

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) = x + \dots. \tag{58}$$