Homework No. 10 (Fall 2021)

PHYS 500A: MATHEMATICAL METHODS

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Due date: Friday, 2021 Dec 3, 4.30pm

The components J_i $(i = 1, 2, 3)$ of angular momentum $\mathbf{J}$ satisfy the commutation relations

$$\frac{1}{i\hbar}\big[J_i, J_j\big] = \varepsilon_{ijk} J_k. \tag{1}$$

The general properties of angular momentum can be deduced from these commutation relations. Since $\mathbf{J}^2$ is a scalar, it commutes with angular momentum $\mathbf{J}$. That is,

$$\big[\mathbf{J}\cdot\mathbf{J}, \mathbf{J}\big] = 0. \tag{2}$$

Thus, the common eigenvectors of $\mathbf{J}^2$ and J_z constitute a suitable set of basis vectors for discussing a dynamical system involving only the angular momentum. Let us denote the eigenvalues of these operators by the labeling scheme $\mathbf{J}'^2 = j(j+1)\hbar^2$, and $J_z{}' = m\hbar$. Thus, we write

$$\frac{1}{\hbar^2}\mathbf{J}^2|j,m\rangle = j(j+1)|j,m\rangle, \tag{3a}$$

$$\frac{1}{\hbar}J_z|j,m\rangle = m|j,m\rangle. \tag{3b}$$

Let us also construct (non-Hermitian) operators

$$J_\pm = J_x \pm iJ_y. \tag{4}$$

Observe that

$$J_\pm^\dagger = J_\mp. \tag{5}$$

1. Show that

$$\frac{1}{\hbar}J_z\Big\{J_+|j,m\rangle\Big\} = (m+1)\Big\{J_+|j,m\rangle\Big\}. \tag{6}$$

Thus deduce that if m is an eigenvalue of J_z, then $(m+1)$ is also an eigenvlaue of J_z. Similarly, show that

$$\frac{1}{\hbar}J_z\Big\{J_-|j,m\rangle\Big\} = (m-1)\Big\{J_-|j,m\rangle\Big\}. \tag{7}$$

Thus deduce that if m is an eigenvalue of J_z, then $(m-1)$ is also an eigenvlaue of J_z.

2. Show that

$$J_+J_- = \mathbf{J}^2 - J_z^2 + \hbar J_z \tag{8}$$

is a Hermitian operator. A Hermitian operator has real eigenvalues, but, since $J_+J_- = J_-^\dagger J_-$, infer further that it has non-negative eigenvalues. Thus, deduce that

$$j(j+1) - m(m-1) \geq 0, \tag{9}$$

and then infer

$$-j \leq m \leq j+1. \tag{10}$$

Similarly, show how that

$$J_-J_+ = \mathbf{J}^2 - J_z^2 - \hbar J_z \tag{11}$$

is a Hermitian operator, and deduce that

$$j(j+1) - m(m+1) \geq 0, \tag{12}$$

and then infer

$$-j-1 \leq m \leq j. \tag{13}$$

Using Eqs. (10) and (13) in conjunction, show that

$$-j \leq m \leq j. \tag{14}$$

(Refer Sec. 36 Dirac's QM book.)

3. Using Eq. (14) infer that

$$j \geq 0. \tag{15}$$

Note that $\mathbf{J}^2$ being the square of a Hermitian operator implies $j(j+1) \geq 0$, but it does not imply that j should be non-negative.

4. The $2j$ transitions from $m=-j$ to $m=j$ happen in $n=0,1,2,\ldots$ steps. Thus, $2j=n$. Thus, conclude that

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \tag{16a}$$

$$m = -j, -j+1, \ldots, j. \tag{16b}$$