Homework No. 09 (Fall 2021)

PHYS 500A: MATHEMATICAL METHODS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Nov 16, 4.30pm

1. (**50 points.**) A quantum harmonic oscillator can be constructed out of two non-Hermitian operators, y and $y^\dagger$, that satisfy the commutation relation

$$[y, y^\dagger] = 1. \tag{1}$$

The eigenstate spectrum of the (Hermitian) number operator, $n = y^\dagger y$, represented by $|n'\rangle$, where $n' = 0, 1, 2, \ldots$, satisfy

$$n|n'\rangle = n'|n'\rangle, \qquad y|n'\rangle = \sqrt{n'}|n'-1\rangle, \qquad y^\dagger|n'\rangle = \sqrt{n'+1}|n'+1\rangle. \tag{2}$$

(a) Build the matrix representation of the lowering operator y using

$$y = \begin{bmatrix} \langle 0|y|0\rangle & \langle 0|y|1\rangle & \langle 0|y|2\rangle & \langle 0|y|3\rangle & \langle 0|y|4\rangle & \cdots \\ \langle 1|y|0\rangle & \langle 1|y|1\rangle & \langle 1|y|2\rangle & \langle 1|y|3\rangle & \langle 1|y|4\rangle & \cdots \\ \langle 2|y|0\rangle & \langle 2|y|1\rangle & \langle 2|y|2\rangle & \langle 2|y|3\rangle & \langle 2|y|4\rangle & \cdots \\ \langle 3|y|0\rangle & \langle 3|y|1\rangle & \langle 3|y|2\rangle & \langle 3|y|3\rangle & \langle 3|y|4\rangle & \cdots \\ \langle 4|y|0\rangle & \langle 4|y|1\rangle & \langle 4|y|2\rangle & \langle 4|y|3\rangle & \langle 4|y|4\rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3}$$

Kindly calculate the first 5×5 block of the infinite dimensional matrix to report the pattern in the following questions.

(b) Similarly, build the matrix representation of the raising operator $y^\dagger$.

(c) Build the matrix representation of the number operator n.

(d) Using the constructions

$$y = \frac{1}{\sqrt{2\hbar}}(x + ip) \quad \text{and} \quad y^\dagger = \frac{1}{\sqrt{2\hbar}}(x - ip), \tag{4}$$

determine the matrix representations for the Hermitian operators, x and p. Check that x and p are indeed Hermitian matrices.

(e) Determine the matrices for the operators xp and px, and verify the commutation relation

$$\frac{1}{i\hbar}[x, p] = 1. \tag{5}$$