Notes on Harmonic oscillator

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1. Any system obeying the Newtonian equation of motion

$$\frac{dp}{dt} = -kx = -\frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right) \tag{1}$$

is broadly termed a simple harmonic oscillator.

2. A harmonic oscillator is also described by the Hamiltonian

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}kx^2. \tag{2}$$

3. We upgrade to quantum mechanics from the classical picture by imposing the commutation relations

$$\big[x,x\big] = 0, \qquad \big[p,p\big] = 0, \qquad \big[x,p\big] = i\hbar, \qquad (x^\dagger = x, \quad p^\dagger = p,) \tag{3}$$

on the dynamical variables x and p. Further, the Hamilton equations of motion

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \tag{4a}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}, \tag{4b}$$

are generalized to include the Heisenberg equations of motion

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{1}{i\hbar}\big[x,H\big], \tag{5a}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = \frac{1}{i\hbar}\big[p,H\big]. \tag{5b}$$

Evaluate the Hamilton and Heisenberg equations of motion for the harmonic oscillator. Further, by projecting the Hamiltonian operator on an arbitrary state $|\,\rangle$, and then projecting the resultant state on a position eigenstate $\langle x'|$, and identifying the expectation value of the Hamiltonian as the energy E of the system in state $|\,\rangle$,

$$\langle x'|H|\,\rangle = E\langle x'|\,\rangle, \tag{6}$$

derive the time-independent Schrödinger equation for the harmonic oscillator,

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2} + \frac{1}{2}kx'^2\right)\psi(x') = E\,\psi(x'), \tag{7}$$

where $\psi(x') = \langle x'|\,\rangle$ is the Schrödinger wave function.

4. In terms of the non-Hermitian operators,

$$y = \frac{1}{\sqrt{2\hbar\omega}} \left(\sqrt{k}\, x + i \frac{p}{\sqrt{m}} \right), \tag{8a}$$

$$y^\dagger = \frac{1}{\sqrt{2\hbar\omega}} \left(\sqrt{k}\, x - i \frac{p}{\sqrt{m}} \right), \tag{8b}$$

where $\omega = \sqrt{k/m}$, verify that

$$\big[y, y^\dagger\big] = 1, \qquad \big[y, y\big] = 0, \qquad \big[y^\dagger, y^\dagger\big] = 0. \tag{9}$$

Show that the Hamiltonian for a harmonic oscillator is given by

$$H(y, y^\dagger) = \hbar\omega \left(y^\dagger y + \frac{1}{2} \right). \tag{10}$$

Evaluate the content of the Heisenberg equations of motion

$$\frac{dy}{dt} = \frac{1}{i\hbar} \big[y, H\big], \tag{11a}$$

$$\frac{dy^\dagger}{dt} = \frac{1}{i\hbar} \big[y^\dagger, H\big]. \tag{11b}$$

5. The Hamiltonian of a harmonic oscillator is given in terms of the operator

$$n = y^\dagger y. \tag{12}$$

Thus, we shall be interested in finding the eigenvalues and eigenvectors of this operator. In particular, our goal will be to solve the eigenvalue equation

$$n|n'\rangle = n'|n'\rangle, \tag{13}$$

where $|n'\rangle$ are the eigenvectors corresponding to the eigenvalues n'.

(a) Argue that $y^\dagger y$ can not be represented using finite dimensional matrices. Thus, argue that the eigenvalues n' must be infinite of them.

(b) Prove that $y^\dagger y$ is an Hermitian operator. Thus, the eigenvalues n' must be real. Further, deduce that the eigenvalues n' are non-negative.

(c) Show that y is a lowering operator, that is,

$$y|n'\rangle = c_{n'}|n' - 1\rangle, \tag{14}$$

where $c_{n'}$ is to be determined. Thus, prove that, if n' is an eigenvalue, then $(n' - 1)$ is also an eigenvalue.

(d) Show that $y^\dagger$ is a raising operator, that is,

$$y^\dagger|n'\rangle = d_{n'}|n'+1\rangle, \tag{15}$$

where $d_{n'}$ is to be determined. Thus, prove that, if n' is an eigenvalue, then $(n'+1)$ is also an eigenvalue.

(e) Together, to satisfy the requirement that the eigenvalues are non-negative, the implication is that the lowering operator can not indefinitely lower the state. Thus, argue the existance of the ground eigenstate that satisfies

$$y|0\rangle = 0. \tag{16}$$

Thus, learn that $n' = 0$ is an eigenstate.

6. To determine $d_{n'}$ we construct an eigenstate in terms of the ground state,

$$|n'\rangle = \frac{(y^\dagger)^{n'}}{D_{n'}}|0\rangle, \qquad D_{n'} = d_0 d_1 \ldots d_{n'-1}. \tag{17}$$

Presuming the eigenstates are normalized, use

$$\langle m'|n'\rangle = \delta_{m'n'} \tag{18}$$

to learn that

$$|D_{n'}|^2 = \langle 0|y^{n'}(y^\dagger)^{n'}|0\rangle. \tag{19}$$

Show that

$$\langle 0|y^{n'}(y^\dagger)^{n'}|0\rangle = n' \,\langle 0|y^{n'-1}(y^\dagger)^{n'-1}|0\rangle. \tag{20}$$

(Hint: Show that $\left[y, (y^\dagger)^{n'}\right] = n'(y^\dagger)^{n'-1}$.) Thus, deduce that $|D_{n'}| = \sqrt{n!}$ and $|d_{n'}| = \sqrt{n'+1}$. Thus, we have

$$y^\dagger|n'\rangle = \sqrt{n'+1}|n'+1\rangle. \tag{21}$$

Operate the lowering operator y on both sides of the above equation and decipher the statement

$$y|n'\rangle = \sqrt{n'}|n'-1\rangle \tag{22}$$

to learn that $c_{n'} = \sqrt{n'}$.

7. The projection of the eigenstates $|n'\rangle$ on the position eigenstates $\langle x'|$ leads to the construction of the corresponding wavefunctions

$$\psi_{n'}(x') = \langle x'|n'\rangle. \tag{23}$$

We will set $k = m = 1$ by suitable scaling of x and p and further set $\hbar = 1$ to avoid clutter in the equations. We shall also drop the primes to represent the eigenvalues.

(a) Starting from Eq. (16) deduce the differential equation satisfied by the ground state to be

$$\left(x + \frac{\partial}{\partial x}\right) \psi_0(x) = 0. \tag{24}$$

Thus, show that

$$\psi_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2}. \tag{25}$$

Plot $\psi_0(x)$.

(b) Starting from the statement in Eq (17),

$$|n\rangle = \frac{(y^\dagger)^n}{\sqrt{n!}} |0\rangle, \tag{26}$$

deduce the relation

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(x - \frac{\partial}{\partial x}\right)^n \psi_0(x). \tag{27}$$

(c) Verify that

$$\left(x - \frac{\partial}{\partial x}\right) f(x) = \left(-e^{\frac{1}{2}x^2} \frac{\partial}{\partial x} e^{-\frac{1}{2}x^2}\right) f(x). \tag{28}$$

Thus, show that

$$\left(x - \frac{\partial}{\partial x}\right)^n f(x) = e^{\frac{1}{2}x^2} \left(-\frac{\partial}{\partial x}\right)^n e^{-\frac{1}{2}x^2} f(x). \tag{29}$$

(d) The Hermite polynomials are defined as

$$H_n(x) = e^{x^2} \left(-\frac{\partial}{\partial x}\right)^n e^{-x^2}. \tag{30}$$

Evaluate $H_n(x)$ for $n = 0, 1, 2, 3, 4$. Thus, derive

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \psi_0(x) H_n(x). \tag{31}$$

Plot $\psi_n(x)$ for $n = 0, 1, 2, 3, 4$.