# Homework No. 08 (Fall 2021)

## PHYS 500A: MATHEMATICAL METHODS

*Department of Physics, Southern Illinois University–Carbondale*

Due date: Tuesday, 2021 Nov 9, 4.30pm

1. (**20 points.**) Consider the eigenvalue equation
$$\sigma_x|\sigma'_x\rangle = \sigma'_x|\sigma'_x\rangle, \tag{1}$$
where primes denote eigenvalues.

   (a) Find the eigenvalues and normalized eigenvectors (up to a phase) of
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$
For reference we shall call these eigenvectors $|\sigma'_x = +\rangle$ and $|\sigma'_x = -\rangle$.

   (b) Now compute the new matrix
$$\bar{\sigma}_x = \begin{pmatrix} \langle\sigma'_x = +|\sigma_x|\sigma'_x = +\rangle & \langle\sigma'_x = +|\sigma_x|\sigma'_x = -\rangle \\ \langle\sigma'_x = -|\sigma_x|\sigma'_x = +\rangle & \langle\sigma'_x = -|\sigma_x|\sigma'_x = -\rangle \end{pmatrix}. \tag{3}$$

   (c) Similarly, compute the new matrices
$$\bar{\sigma}_y = \begin{pmatrix} \langle\sigma'_x = +|\sigma_y|\sigma'_x = +\rangle & \langle\sigma'_x = +|\sigma_y|\sigma'_x = -\rangle \\ \langle\sigma'_x = -|\sigma_y|\sigma'_x = +\rangle & \langle\sigma'_x = -|\sigma_y|\sigma'_x = -\rangle \end{pmatrix}, \quad \text{where} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{4}$$
and
$$\bar{\sigma}_z = \begin{pmatrix} \langle\sigma'_x = +|\sigma_z|\sigma'_x = +\rangle & \langle\sigma'_x = +|\sigma_z|\sigma'_x = -\rangle \\ \langle\sigma'_x = -|\sigma_z|\sigma'_x = +\rangle & \langle\sigma'_x = -|\sigma_z|\sigma'_x = -\rangle \end{pmatrix}, \quad \text{where} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

   (d) Find the product of the last two matrices, $\bar{\sigma}_y\bar{\sigma}_z$, and express it in terms of $\bar{\sigma}_x$.

2. (**20 points.**) The Pauli matrix
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6}$$
is written in the eigenbasis of
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$
Write $\sigma_x$ in the eigenbasis of
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{8}$$
Note that this representation has the arbitraryness of the choice of phase in the eigenvectors.

3. (**20 points.**) Construct the matrix

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}, \tag{9}$$

where

$$\begin{aligned} \boldsymbol{\sigma} &= \sigma_x \hat{\mathbf{i}} + \sigma_y \hat{\mathbf{j}} + \sigma_z \hat{\mathbf{k}}, && (10) \\ \hat{\mathbf{r}} &= \sin\theta \cos\phi \hat{\mathbf{i}} + \sin\theta \sin\phi \hat{\mathbf{j}} + \cos\theta \hat{\mathbf{k}}. && (11) \end{aligned}$$

Use the following representation of Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}$$

Find the eigenvalues of the matrix $\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}$.

4. (**20 points.**) Consider a normalized state

$$|\,\rangle = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x_1 + i x_2 \\ y_1 + i y_2 \end{pmatrix}, \qquad |u|^2 + |v|^2 = 1. \tag{13}$$

(a) Show that the expectation value of the Pauli matrices with respect to the above state satisfies

$$\langle \sigma_x \rangle^2 + \langle \sigma_y \rangle^2 + \langle \sigma_z \rangle^2 = 1. \tag{14}$$

Thus, conclude that the magnitude of the expectation value of each of the Pauli matrix is less than or equal to unity.

(b) Define the errors in the measurement of the Pauli matrices to be

$$\begin{aligned} (\delta\sigma'_x)^2 &= \langle\,|\{\sigma_x - \langle\sigma_x\rangle\}^2|\,\rangle, && (15a) \\ (\delta\sigma'_y)^2 &= \langle\,|\{\sigma_y - \langle\sigma_y\rangle\}^2|\,\rangle, && (15b) \\ (\delta\sigma'_z)^2 &= \langle\,|\{\sigma_z - \langle\sigma_z\rangle\}^2|\,\rangle. && (15c) \end{aligned}$$

Show that

$$\begin{aligned} (\delta\sigma'_x)^2 &= 1 - \langle\sigma_x\rangle^2, && (16a) \\ (\delta\sigma'_y)^2 &= 1 - \langle\sigma_y\rangle^2, && (16b) \\ (\delta\sigma'_z)^2 &= 1 - \langle\sigma_z\rangle^2. && (16c) \end{aligned}$$

Thus, conclude that the errors in the measurement of each of the Pauli matrix is less than or equal to unity. Show that

$$(\delta\sigma'_x)^2 + (\delta\sigma'_y)^2 + (\delta\sigma'_z)^2 = 2. \tag{17}$$

(c) Using Robertson's generalization of Heisenberg's uncertainty relation

$$(\delta A)(\delta B) \geq \frac{1}{2}|\langle C\rangle|, \qquad C = \frac{1}{i}\big[A, B\big], \tag{18}$$

deduce the uncertainty relations for the Pauli matrices to be

$$(\delta\sigma'_x)^2(\delta\sigma'_y)^2 \geq \langle\sigma_z\rangle^2, \tag{19a}$$
$$(\delta\sigma'_y)^2(\delta\sigma'_z)^2 \geq \langle\sigma_x\rangle^2, \tag{19b}$$
$$(\delta\sigma'_z)^2(\delta\sigma'_x)^2 \geq \langle\sigma_y\rangle^2. \tag{19c}$$

Combine these uncertainty relations to derive an uncertainty relation involving all the three Pauli matrices,

$$(\delta\sigma'_x)^2(\delta\sigma'_y)^2 + (\delta\sigma'_y)^2(\delta\sigma'_z)^2 + (\delta\sigma'_z)^2(\delta\sigma'_x)^2 \geq 1. \tag{20}$$