Homework No. 05 (Fall 2021)

PHYS 500A: MATHEMATICAL METHODS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Sep 28, 4.30pm

1. (**20 points.**) Recall that analytic functions satisfy the Cauchy-Riemann equations. That is, the real and imaginary parts of an analytic function

$$f(x+iy) = u(x,y) + iv(x,y) \tag{1}$$

satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{2a}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{2b}$$

Given $f(z)$ and $g(z)$ are analytic functions in a region, then show that $f(g(z))$ satisfies the Cauchy-Riemann equations there.

Hint: Let $g = u + iv$ and $f = U + iV$. Thus, we can write

$$f(g(z)) = U(u(x,y), v(x,y)) + iV(u(x,y), v(x,y)). \tag{3}$$

2. (**20 points.**) Given an analytic function

$$f(z) = u(x,y) + iv(x,y) \tag{4}$$

and the gradient operator

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}. \tag{5}$$

Show that Cauchy-Riemann equations imply

$$(\boldsymbol{\nabla} u) \cdot (\boldsymbol{\nabla} v) = 0. \tag{6}$$

Thus, interpret that u's and v's are orthogonal family of surfaces at every point.

3. (**20 points.**) Check if the function

$$f(z) = e^z + e^{iz} \tag{7}$$

satisfies the Cauchy-Riemann conditions. If $f(z)$ is analytic for all z, then report the derivative as a function of z. Otherwise, determine the points, or regions, in the z plane where the function is not analytic.

4. (**20 points.**) Check if the function

$$f(z) = \frac{1}{z} \tag{8}$$

satisfies the Cauchy-Riemann conditions.

(a) Verify that the Cauchy-Riemann conditions for this case are not well defined at $z = 0$, but are fine for $z \neq 0$.

(b) Verify that

$$\frac{df}{dz} = -\frac{1}{z^2}, \qquad z \neq 0. \tag{9}$$

(c) Determine the limiting value of the derivative as you approach $z = 0$ along the positive real line, and, then, when you approach along the negative real line. Repeat the analysis along the imaginary line. Repeat the analysis along the line $x = y$. Are these limits identical?

(d) If these limits are not identical conclude that the derivative is not isotropic at $z = 0$. Then, the function is not analytic at $z = 0$.