Homework No. 04 (Fall 2021)

PHYS 500A: MATHEMATICAL METHODS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Sep 21, 4.30pm

1. (**20 points.**) Verify that
$$\sqrt{-2}\sqrt{-3} = -\sqrt{6}. \tag{1}$$
It is recommended to write $\sqrt{2}e^{i\frac{\pi}{2}}$ in place of $\sqrt{-2}$. This is because it is tempting to incorrectly conclude $\sqrt{-2}\sqrt{-3} = \sqrt{(-2)(-3)} = \sqrt{6}$. The associated error was not convincing and led to confusions until the geometric visualization of a complex number using Argand diagram (magnitude and direction in polar representation) was invented by Wessel in 1797 and popularized by Argand in 1806. This geometric interpretation led to an explosion of ideas in the field of complex numbers. Without this geometric interpretation even Euler fell into the trap of concluding erroneously $\sqrt{-2}\sqrt{-3} = \sqrt{6}$.

2. (**20 points.**) Evaluate
$$(3 + i4)^{23}. \tag{2}$$
Mark the resulting number on the complex plane.

3. (**20 points.**) Prove the identity
$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}. \tag{3}$$
Hint: $(2 + i)(3 + i) = 5 + i5$.

4. (**20 points.**) The close connection between the geometry of a complex number
$$z = x + iy \tag{4}$$
and a two-dimensional vector
$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} \tag{5}$$
is intriguing. They have the same rules for addition and subtraction, but differ in their rules for multiplication. Show that
$$z_1^* z_2 = (\mathbf{r}_1 \cdot \mathbf{r}_2) + i(\mathbf{r}_1 \times \mathbf{r}_2) \cdot \hat{\mathbf{k}}. \tag{6}$$
In the quest for a number system that corresponds to a three dimensional vector, Hamilton in 1843 invented the quaternions. A quaternion P can be expressed in terms of Pauli matrices as
$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}. \tag{7}$$
Recall that the Pauli matrices are completely characterized by the identity
$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b}) + i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{8}$$

(a) Show that the (Hamilton) product of two quaternions,

$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}, \tag{9a}$$

$$Q = b_0 - i\mathbf{b} \cdot \boldsymbol{\sigma}, \tag{9b}$$

is given by

$$PQ = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) - i(a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{10}$$

(b) Verify that the Hamilton product is non-commutative. Determine $\big[P, Q\big]$.