Homework No. 02 (Fall 2021)

PHYS 500A: MATHEMATICAL METHODS

Department of Physics, Southern Illinois University–Carbondale

Due date: Monday, 2021 Aug 30, 4.30pm

1. (**Example.**) Let $\mathbf{r}$ represent a position vector in three dimensional space. Let x^i be the components of the position vector in rectangular coordinates, which can be interpreted as surfaces of constant x^i. Let us coordinatize the space using the planes, labeled using β,

$$y = mx + \beta \tag{1}$$

where m is fixed, instead of planes with constant y. The other two sets of planes of constant x and constant z are the same. See Fig. 1. Let u^i be the components of the position vector in this new coordinatization of space. In particular, we have

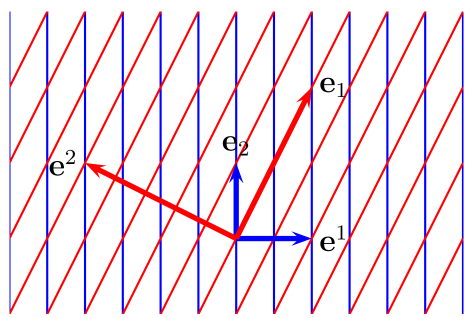

Figure 1: Basis vectors $\mathbf{e}_i$ and reciprocal basis vectors $\mathbf{e}^i$.

$$x^1 = x = \alpha, \qquad u^1 = \alpha = x, \tag{2a}$$
$$x^2 = y = mx + \beta, \qquad u^2 = \beta = y - mx, \tag{2b}$$
$$x^3 = z = \gamma, \qquad u^3 = \gamma = z. \tag{2c}$$

The basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ in rectangular coordinate system will be represented as $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, if necessary.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{3}$$

Show that

$$\mathbf{e}_1 = \hat{\mathbf{i}} + m\,\hat{\mathbf{j}}, \qquad \mathbf{e}_2 = \hat{\mathbf{j}}, \qquad \mathbf{e}_3 = \hat{\mathbf{k}}. \tag{4}$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{5}$$

Show that

$$\mathbf{e}^1 = \hat{\mathbf{i}}, \qquad \mathbf{e}^2 = -m\,\hat{\mathbf{i}} + \hat{\mathbf{j}}, \qquad \mathbf{e}^3 = \hat{\mathbf{k}}. \tag{6}$$

Verify the relations

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_2 \times \mathbf{e}_3)\cdot \mathbf{e}_1}, \qquad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{(\mathbf{e}_3 \times \mathbf{e}_1)\cdot \mathbf{e}_2}, \qquad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{(\mathbf{e}_1 \times \mathbf{e}_2)\cdot \mathbf{e}_3}. \tag{7}$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{8}$$

That is,

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = 1, \qquad \mathbf{e}^1 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}^1 \cdot \mathbf{e}_3 = 0, \tag{9a}$$

$$\mathbf{e}^2 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}^2 \cdot \mathbf{e}_2 = 1, \qquad \mathbf{e}^2 \cdot \mathbf{e}_3 = 0, \tag{9b}$$

$$\mathbf{e}^3 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}^3 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}^3 \cdot \mathbf{e}_3 = 1. \tag{9c}$$

(d) Metric tensor: The metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{10}$$

Evaluate all the components of g_{ij}. That is,

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1 + m^2, \qquad g_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2 = m, \qquad g_{13} = \mathbf{e}_1 \cdot \mathbf{e}_3 = 0, \tag{11a}$$

$$g_{21} = \mathbf{e}_2 \cdot \mathbf{e}_1 = m, \qquad g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1, \qquad g_{23} = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0, \tag{11b}$$

$$g_{31} = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \qquad g_{32} = \mathbf{e}_3 \cdot \mathbf{e}_2 = 0, \qquad g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1. \tag{11c}$$

Similarly evaluate the components of

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j. \tag{12}$$

That is,

$$g^{11} = \mathbf{e}^1 \cdot \mathbf{e}^1 = 1, \qquad g^{12} = \mathbf{e}^1 \cdot \mathbf{e}^2 = -m, \qquad g^{13} = \mathbf{e}^1 \cdot \mathbf{e}^3 = 0, \tag{13a}$$

$$g^{21} = \mathbf{e}^2 \cdot \mathbf{e}^1 = -m, \qquad g^{22} = \mathbf{e}^2 \cdot \mathbf{e}^2 = 1 + m^2, \qquad g^{23} = \mathbf{e}^2 \cdot \mathbf{e}^3 = 0, \tag{13b}$$

$$g^{31} = \mathbf{e}^3 \cdot \mathbf{e}^1 = 0, \qquad g^{32} = \mathbf{e}^3 \cdot \mathbf{e}^2 = 0, \qquad g^{33} = \mathbf{e}^3 \cdot \mathbf{e}^3 = 1. \tag{13c}$$

Verify that $g^{ij} g_{jk} = \delta^i_k$.

(e) Completeness relation: Verify the completeness relation

$$\mathbf{e}^i \mathbf{e}_i = \mathbf{1} \tag{14}$$

by evaluating

$$\mathbf{e}^1 \mathbf{e}_1 + \mathbf{e}^2 \mathbf{e}_2 + \mathbf{e}^3 \mathbf{e}_3. \tag{15}$$

(f) Given a vector

$$\mathbf{A} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}} \tag{16}$$

in rectangular coordinates, find the components of the vector $\mathbf{A}$ in the basis of $\mathbf{e}_i$. That is, find the components A^i in

$$\mathbf{A} = A^1\,\mathbf{e}_1 + A^2\,\mathbf{e}_2 + A^3\,\mathbf{e}_3. \tag{17}$$