Midterm Exam No. 03 (Fall 2021)

PHYS 500A: MATHEMATICAL METHODS

Department of Physics, Southern Illinois University–Carbondale

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1. (**20 points.**) The Pauli matrix

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1}$$

is written in the eigenbasis of

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Write σ_y in the eigenbasis of

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

Note that this representation has the arbitraryness of the choice of phase in the eigenvectors.

2. (**20 points.**) In the eigenbasis,

$$|\sigma'_y = +\rangle = \frac{e^{i\alpha_1}}{\sqrt{2}} \begin{pmatrix} i \\ -1 \end{pmatrix} \quad \text{and} \quad |\sigma'_y = -\rangle = \frac{e^{i\alpha_2}}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}, \tag{4}$$

of

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{5}$$

where we included the arbitrariness in the phases as α_1 and α_2, the Pauli matrices are

$$\bar{\sigma}_x = \begin{pmatrix} 0 & -ie^{-i(\alpha_1-\alpha_2)} \\ ie^{i(\alpha_1-\alpha_2)} & 0 \end{pmatrix}, \tag{6a}$$

$$\bar{\sigma}_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6b}$$

$$\bar{\sigma}_z = \begin{pmatrix} 0 & e^{-i(\alpha_1-\alpha_2)} \\ e^{i(\alpha_1-\alpha_2)} & 0 \end{pmatrix}. \tag{6c}$$

Thus, these representations of Pauli matrices depend on the arbitrary choice of phases. (No need to derive the above here.) Do we loose the ability to predict the outcome of an experiment due to this arbitrariness? Verify that

$$\bar{\sigma}_x \bar{\sigma}_y = i\bar{\sigma}_z, \tag{7a}$$

$$\bar{\sigma}_y \bar{\sigma}_z = i\bar{\sigma}_x, \tag{7b}$$

$$\bar{\sigma}_z \bar{\sigma}_x = i\bar{\sigma}_y. \tag{7c}$$

Thus, the algebra of the Pauli matrices is independent of the arbitrariness in the phases. This will ensure that no measurable quantity depends on the choice of phases.

3. (**20 points.**) Consider an operator U that is defined using the following operations,

$$U|a_1\rangle = |a_2\rangle, \tag{8a}$$

$$U|a_2\rangle = |a_3\rangle, \tag{8b}$$

$$U|a_3\rangle = |a_1\rangle. \tag{8c}$$

   Find the eigenvalues of the operator U.

4. (**20 points.**) Using the commutation relations involving the position vector $\mathbf{r}$, the linear momentum vector $\mathbf{p}$, and the orbital angular momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, show that

$$\frac{1}{i\hbar}\big(\mathbf{p} \times \mathbf{L} + \mathbf{L} \times \mathbf{p}\big) = a\mathbf{p}, \tag{9}$$

   where a is a number. Report the numerical value for a.
   Caution: Using index notation might be less error prone here.