

Final Exam (Fall 2021)

PHYS 500A: MATHEMATICAL METHODS

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1. (10 points.) Using commutation relations between position \mathbf{r} and momentum \mathbf{p} , verify the relation

$$\mathbf{r} \cdot \mathbf{p} - \mathbf{p} \cdot \mathbf{r} = c i \hbar, \quad (1)$$

where c is a number. Determine c .

2. (10 points.) Show that if $f(z)$ is an analytic function then the derivative df/dz satisfies the Cauchy-Riemann equations.
3. (10 points.) Consider an operator U that is defined using the following operations,

$$U|a_1\rangle = |a_2\rangle, \quad (2a)$$

$$U|a_2\rangle = |a_3\rangle, \quad (2b)$$

$$U|a_3\rangle = |a_4\rangle. \quad (2c)$$

$$U|a_4\rangle = |a_1\rangle. \quad (2d)$$

Find the eigenvalues of the operator U .

4. (70 points.) The angular momentum can be decomposed as

$$\mathbf{J} = \mathbf{S} + \mathbf{L}, \quad (3)$$

where \mathbf{S} is the spin or internal angular momentum, and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the orbital or external angular momentum. For the case $\mathbf{S} = 0$ the eigenvalues of angular momentum are necessarily integer valued, because $\mathbf{r} \cdot \mathbf{L} = 0$. Let us denote the eigenvalues by the labeling scheme $\mathbf{L}^2 = \hbar^2 l(l+1)$ and $L'_z = \hbar m$, such that

$$\mathbf{L}^2|l, m\rangle = \hbar^2 l(l+1)|l, m\rangle, \quad (4a)$$

$$L'_z|l, m\rangle = \hbar m|l, m\rangle, \quad (4b)$$

where

$$l = 0, 1, 2, \dots, \quad (5a)$$

$$m = -l, -l+1, \dots, l. \quad (5b)$$

The eigenvectors of orbital angular momentum are suitably realized by functions on the surface of a unit sphere, coordinated by spherical polar coordinates θ' and ϕ' or the unit vector $\hat{\mathbf{r}}'$. These wavefunctions defined using the projections

$$\langle \hat{\mathbf{r}}'|l, m\rangle = Y_{lm}(\theta', \phi') \quad (6)$$

are the spherical harmonics.

- (a) Show that in the position basis, here restricted to the surface of a unit sphere, we have

$$\langle \hat{\mathbf{r}}' | \mathbf{L} | \rangle = \langle \hat{\mathbf{r}}' | \mathbf{r} \times \mathbf{p} | \rangle = \frac{\hbar}{i} (\mathbf{r}' \times \nabla') \langle \hat{\mathbf{r}}' | \rangle. \quad (7)$$

Using Eq. (7) in Eqs. (4) show that the differential equations for spherical harmonics are given by

$$-(\mathbf{r}' \times \nabla') \cdot (\mathbf{r}' \times \nabla') Y_{lm}(\theta', \phi') = l(l+1) Y_{lm}(\theta', \phi'), \quad (8a)$$

$$\frac{1}{i} \hat{\mathbf{z}}' \cdot (\mathbf{r}' \times \nabla') Y_{lm}(\theta', \phi') = m Y_{lm}(\theta', \phi'). \quad (8b)$$

- (b) Show that the raising and lowering operators defined using

$$L_{\pm} = L_x \pm iL_y, \quad (9)$$

leading to raising and lowering operations

$$L_{\pm} |l, m\rangle = \hbar \sqrt{(l \mp m)(l \pm m + 1)} |l, m \pm 1\rangle, \quad (10)$$

correspond to the differential equations

$$\frac{1}{i} \left[\hat{\mathbf{x}}' \cdot (\mathbf{r}' \times \nabla') \pm i \hat{\mathbf{y}}' \cdot (\mathbf{r}' \times \nabla') \right] Y_{lm}(\theta', \phi') = \sqrt{(l \mp m)(l \pm m + 1)} Y_{l, m \pm 1}(\theta', \phi'). \quad (11)$$

- (c) Using the differential operator in spherical polar coordinates,

$$\nabla' = \hat{\mathbf{r}}' \frac{\partial}{\partial r'} + \hat{\boldsymbol{\theta}}' \frac{1}{r'} \frac{\partial}{\partial \theta'} + \hat{\boldsymbol{\phi}}' \frac{1}{r' \sin \theta'} \frac{\partial}{\partial \phi'}, \quad (12)$$

where

$$\hat{\mathbf{r}}' = \hat{\mathbf{x}}' \sin \theta' \cos \phi' + \hat{\mathbf{y}}' \sin \theta' \sin \phi' + \hat{\mathbf{z}}' \cos \theta', \quad (13a)$$

$$\hat{\boldsymbol{\theta}}' = \hat{\mathbf{x}}' \cos \theta' \cos \phi' + \hat{\mathbf{y}}' \cos \theta' \sin \phi' - \hat{\mathbf{z}}' \sin \theta', \quad (13b)$$

$$\hat{\boldsymbol{\phi}}' = -\hat{\mathbf{x}}' \sin \phi' + \hat{\mathbf{y}}' \cos \phi', \quad (13c)$$

show that

$$\mathbf{r}' \times \nabla' = \hat{\boldsymbol{\phi}}' \frac{\partial}{\partial \theta'} - \hat{\boldsymbol{\theta}}' \frac{1}{\sin \theta'} \frac{\partial}{\partial \phi'} \quad (14a)$$

$$= \hat{\mathbf{x}}' \left[-\sin \phi' \frac{\partial}{\partial \theta'} - \cos \phi' \cot \theta' \frac{\partial}{\partial \phi'} \right] + \hat{\mathbf{y}}' \left[\cos \phi' \frac{\partial}{\partial \theta'} - \sin \phi' \cot \theta' \frac{\partial}{\partial \phi'} \right] + \hat{\mathbf{z}}' \frac{\partial}{\partial \phi'}. \quad (14b)$$

Thus, show the correspondence

$$L_z : \quad \hat{\mathbf{z}}' \cdot \frac{\hbar}{i} (\mathbf{r}' \times \nabla') = \frac{\hbar}{i} \frac{\partial}{\partial \phi'}, \quad (15a)$$

$$L_x : \quad \hat{\mathbf{x}}' \cdot \frac{\hbar}{i} (\mathbf{r}' \times \nabla') = \frac{\hbar}{i} \left[-\sin \phi' \frac{\partial}{\partial \theta'} - \cos \phi' \cot \theta' \frac{\partial}{\partial \phi'} \right], \quad (15b)$$

$$L_y : \quad \hat{\mathbf{y}}' \cdot \frac{\hbar}{i} (\mathbf{r}' \times \nabla') = \frac{\hbar}{i} \left[\cos \phi' \frac{\partial}{\partial \theta'} - \sin \phi' \cot \theta' \frac{\partial}{\partial \phi'} \right]. \quad (15c)$$

Further, verify the correspondence

$$L^2 : \quad \frac{\hbar}{i}(\mathbf{r}' \times \nabla') \cdot \frac{\hbar}{i}(\mathbf{r}' \times \nabla') = \frac{\hbar^2}{i^2} \left[\frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \sin \theta' \frac{\partial}{\partial \theta'} + \frac{1}{\sin^2 \theta'} \frac{\partial^2}{\partial \phi'^2} \right], \quad (16a)$$

$$L_z^2 : \quad \frac{\hbar^2}{i^2} \left[\hat{\mathbf{z}}' \cdot (\mathbf{r}' \times \nabla') \right]^2 = \frac{\hbar^2}{i^2} \frac{\partial^2}{\partial \phi'^2}, \quad (16b)$$

$$L_{\pm} : \quad \frac{\hbar}{i} \left[\hat{\mathbf{x}}' \cdot (\mathbf{r}' \times \nabla') \pm i \hat{\mathbf{y}}' \cdot (\mathbf{r}' \times \nabla') \right] = \frac{\hbar}{i} e^{\pm i\phi} \left[\pm i \frac{\partial}{\partial \theta'} - \cot \theta' \frac{\partial}{\partial \phi'} \right], \quad (16c)$$

(d) Thus, show that the eigenfunctions of angular momentum in the position basis, the spherical harmonics, satisfy the differential equations given by

$$L_z : \quad \frac{1}{i} \frac{\partial}{\partial \phi'} Y_{lm}(\theta', \phi') = m Y_{lm}(\theta', \phi'), \quad (17a)$$

$$L^2 : \quad - \left[\frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \sin \theta' \frac{\partial}{\partial \theta'} + \frac{1}{\sin^2 \theta'} \frac{\partial^2}{\partial \phi'^2} \right] Y_{lm}(\theta', \phi') = l(l+1) Y_{lm}(\theta', \phi'), \quad (17b)$$

$$L_{\pm} : \quad \frac{i}{i} e^{\pm i\phi} \left[\pm i \frac{\partial}{\partial \theta'} - \cot \theta' \frac{\partial}{\partial \phi'} \right] Y_{lm}(\theta', \phi') = \sqrt{(l \mp m)(l \pm m + 1)} Y_{l, m \pm 1}(\theta', \phi'). \quad (17c)$$

Further, verify

$$\begin{aligned} L_+ L_- : \quad & - \left[\frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \sin \theta' \frac{\partial}{\partial \theta'} + \frac{1}{\sin^2 \theta'} \frac{\partial^2}{\partial \phi'^2} - \frac{\partial^2}{\partial \phi'^2} - \frac{1}{i} \frac{\partial}{\partial \phi'} \right] Y_{lm}(\theta', \phi') \\ & = [l(l+1) - m(m-1)] Y_{lm}(\theta', \phi'), \end{aligned} \quad (18a)$$

$$\begin{aligned} L_- L_+ : \quad & - \left[\frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \sin \theta' \frac{\partial}{\partial \theta'} + \frac{1}{\sin^2 \theta'} \frac{\partial^2}{\partial \phi'^2} - \frac{\partial^2}{\partial \phi'^2} + \frac{1}{i} \frac{\partial}{\partial \phi'} \right] Y_{lm}(\theta', \phi') \\ & = [l(l+1) - m(m+1)] Y_{lm}(\theta', \phi'), \end{aligned} \quad (18b)$$

$$\begin{aligned} L_x^2 + L_y^2 : \quad & - \left[\frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \sin \theta' \frac{\partial}{\partial \theta'} + \cot^2 \theta' \frac{\partial^2}{\partial \phi'^2} \right] Y_{lm}(\theta', \phi') \\ & = [l(l+1) - m^2] Y_{lm}(\theta', \phi'). \end{aligned} \quad (18c)$$