(Take-Home part of) Final Exam (Fall 2021)

PHYS 500A: MATHEMATICAL METHODS

Department of Physics, Southern Illinois University–Carbondale

Date: 2021 Dec 10, 8am

1. (**70 points.**) The angular momentum can be decomposed as

$$\mathbf{J} = \mathbf{S} + \mathbf{L}, \tag{1}$$

where $\mathbf{S}$ is the spin or internal angular momentum, and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the orbital or external angular momentum. For the case $\mathbf{S} = 0$ the eigenvalues of angular momentum are necessarily integer valued, because $\mathbf{r} \cdot \mathbf{L} = 0$. Let us denote the eigenvalues by the labeling scheme $\mathbf{L}'^2 = \hbar^2 l(l+1)$ and $L'_z = \hbar m$, such that

$$\mathbf{L}^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle, \tag{2a}$$

$$L_z |l, m\rangle = \hbar m |l, m\rangle, \tag{2b}$$

where

$$l = 0, 1, 2, \ldots, \tag{3a}$$

$$m = -l, -l+1, \ldots, l. \tag{3b}$$

The eigenvectors of orbital angular momentum are suitably realized by functions on the surface of a unit sphere, coordinated by spherical polar coordinates θ' and ϕ' or the unit vector $\hat{\mathbf{r}}'$. These wavefunctions defined using the projections

$$\langle \hat{\mathbf{r}}' | l, m \rangle = Y_{lm}(\theta', \phi') \tag{4}$$

are the spherical harmonics.

(a) Show that in the position basis, here restricted to the surface of a unit sphere, we have

$$\langle \hat{\mathbf{r}}' | \mathbf{L} | \ \rangle = \langle \hat{\mathbf{r}}' | \mathbf{r} \times \mathbf{p} | \ \rangle = \frac{\hbar}{i} (\mathbf{r}' \times \boldsymbol{\nabla}') \langle \hat{\mathbf{r}}' | \ \rangle. \tag{5}$$

Using Eq. (5) in Eqs. (2) show that the differential equations for spherical harmonics are given by

$$-(\mathbf{r}' \times \boldsymbol{\nabla}') \cdot (\mathbf{r}' \times \boldsymbol{\nabla}') Y_{lm}(\theta', \phi') = l(l+1) Y_{lm}(\theta', \phi'), \tag{6a}$$

$$\frac{1}{i} \hat{\mathbf{z}}' \cdot (\mathbf{r}' \times \boldsymbol{\nabla}') Y_{lm}(\theta', \phi') = m Y_{lm}(\theta', \phi'). \tag{6b}$$

(b) Show that the raising and lowering operators defined using

$$L_\pm = L_x \pm i L_y, \tag{7}$$

leading to raising and lowering operations

$$L_\pm |l, m\rangle = \hbar \sqrt{(l \mp m)(l \pm m + 1)} |l, m \pm 1\rangle, \tag{8}$$

correspond to the differential equations

$$\frac{1}{i}\Big[\hat{\mathbf{x}}' \cdot (\mathbf{r}' \times \boldsymbol{\nabla}') \pm i \hat{\mathbf{y}}' \cdot (\mathbf{r}' \times \boldsymbol{\nabla}')\Big] Y_{lm}(\theta', \phi') = \sqrt{(l \mp m)(l \pm m + 1)} Y_{l,m\pm 1}(\theta', \phi'). \tag{9}$$

(c) Using the differential operator in spherical polar coordinates,

$$\boldsymbol{\nabla}' = \hat{\mathbf{r}}' \frac{\partial}{\partial r'} + \hat{\boldsymbol{\theta}}' \frac{1}{r'} \frac{\partial}{\partial \theta'} + \hat{\boldsymbol{\phi}}' \frac{1}{r' \sin\theta'} \frac{\partial}{\partial \phi'}, \tag{10}$$

where

$$\hat{\mathbf{r}}' = \hat{\mathbf{x}}' \sin\theta' \cos\phi' + \hat{\mathbf{y}}' \sin\theta' \sin\phi' + \hat{\mathbf{z}}' \cos\theta', \tag{11a}$$

$$\hat{\boldsymbol{\theta}}' = \hat{\mathbf{x}}' \cos\theta' \cos\phi' + \hat{\mathbf{y}}' \cos\theta' \sin\phi' - \hat{\mathbf{z}}' \sin\theta', \tag{11b}$$

$$\hat{\boldsymbol{\phi}}' = -\hat{\mathbf{x}}' \sin\phi' + \hat{\mathbf{y}}' \cos\phi', \tag{11c}$$

show that

$$\mathbf{r}' \times \boldsymbol{\nabla}' = \hat{\boldsymbol{\phi}}' \frac{\partial}{\partial \theta'} - \hat{\boldsymbol{\theta}}' \frac{1}{\sin\theta'} \frac{\partial}{\partial \phi'} \tag{12a}$$

$$= \hat{\mathbf{x}}' \left[-\sin\phi' \frac{\partial}{\partial \theta'} - \cos\phi' \cot\theta' \frac{\partial}{\partial \phi'} \right] + \hat{\mathbf{y}}' \left[\cos\phi' \frac{\partial}{\partial \theta'} - \sin\phi' \cot\theta' \frac{\partial}{\partial \phi'} \right] + \hat{\mathbf{z}}' \frac{\partial}{\partial \phi'}. \tag{12b}$$

Thus, show the correspondence

$$L_z : \quad \hat{\mathbf{z}}' \cdot \frac{\hbar}{i} (\mathbf{r}' \times \boldsymbol{\nabla}') = \frac{\hbar}{i} \frac{\partial}{\partial \phi'}, \tag{13a}$$

$$L_x : \quad \hat{\mathbf{x}}' \cdot \frac{\hbar}{i} (\mathbf{r}' \times \boldsymbol{\nabla}') = \frac{\hbar}{i} \left[-\sin\phi' \frac{\partial}{\partial \theta'} - \cos\phi' \cot\theta' \frac{\partial}{\partial \phi'} \right], \tag{13b}$$

$$L_y : \quad \hat{\mathbf{y}}' \cdot \frac{\hbar}{i} (\mathbf{r}' \times \boldsymbol{\nabla}') = \frac{\hbar}{i} \left[\cos\phi' \frac{\partial}{\partial \theta'} - \sin\phi' \cot\theta' \frac{\partial}{\partial \phi'} \right]. \tag{13c}$$

Further, verify the correspondence

$$L^2 : \quad \frac{\hbar}{i} (\mathbf{r}' \times \boldsymbol{\nabla}') \cdot \frac{\hbar}{i} (\mathbf{r}' \times \boldsymbol{\nabla}') = \frac{\hbar^2}{i^2} \left[\frac{1}{\sin\theta'} \frac{\partial}{\partial \theta'} \sin\theta' \frac{\partial}{\partial \theta'} + \frac{1}{\sin^2\theta'} \frac{\partial^2}{\partial \phi'^2} \right], \tag{14a}$$

$$L_z^2 : \quad \frac{\hbar^2}{i^2} \Big[\hat{\mathbf{z}}' \cdot (\mathbf{r}' \times \boldsymbol{\nabla}') \Big]^2 = \frac{\hbar^2}{i^2} \frac{\partial^2}{\partial \phi'^2}, \tag{14b}$$

$$L_\pm : \quad \frac{\hbar}{i} \Big[\hat{\mathbf{x}}' \cdot (\mathbf{r}' \times \boldsymbol{\nabla}') \pm i \hat{\mathbf{y}}' \cdot (\mathbf{r}' \times \boldsymbol{\nabla}') \Big] = \frac{\hbar}{i} e^{\pm i\phi} \left[\pm i \frac{\partial}{\partial \theta'} - \cot\theta' \frac{\partial}{\partial \phi'} \right], \tag{14c}$$

(d) Thus, show that the eigenfunctions of angular momentum in the position basis, the spherical harmonics, satisfy the differential equations given by

$$L_z: \qquad \frac{1}{i}\frac{\partial}{\partial\phi'}Y_{lm}(\theta',\phi') = mY_{lm}(\theta',\phi'), \tag{15a}$$

$$L^2: \quad -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2}\right]Y_{lm}(\theta',\phi') = l(l+1)Y_{lm}(\theta',\phi'), \tag{15b}$$

$$L_\pm: \qquad \frac{i}{i}e^{\pm i\phi}\left[\pm i\frac{\partial}{\partial\theta'} - \cot\theta'\frac{\partial}{\partial\phi'}\right]Y_{lm}(\theta',\phi') = \sqrt{(l\mp m)(l\pm m+1)}Y_{l,m\pm1}(\theta',\phi'). \tag{15c}$$

Further, verify

$$\begin{aligned} L_+L_-: \quad & -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2} - \frac{\partial^2}{\partial\phi'^2} - \frac{1}{i}\frac{\partial}{\partial\phi'}\right]Y_{lm}(\theta',\phi') \\ & \qquad = \big[l(l+1) - m(m-1)\big]Y_{lm}(\theta',\phi'), \end{aligned} \tag{16a}$$

$$\begin{aligned} L_-L_+: \quad & -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin^2\theta'}\frac{\partial^2}{\partial\phi'^2} - \frac{\partial^2}{\partial\phi'^2} + \frac{1}{i}\frac{\partial}{\partial\phi'}\right]Y_{lm}(\theta',\phi') \\ & \qquad = \big[l(l+1) - m(m+1)\big]Y_{lm}(\theta',\phi'), \end{aligned} \tag{16b}$$

$$\begin{aligned} L_x^2 + L_y^2: \quad & -\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \cot^2\theta'\frac{\partial^2}{\partial\phi'^2}\right]Y_{lm}(\theta',\phi') \\ & \qquad = \big[l(l+1) - m^2\big]Y_{lm}(\theta',\phi'). \end{aligned} \tag{16c}$$

2. (**10 points.**) Question on vector calculus. To be completed during the exam.

3. (**10 points.**) Question on complex analysis. To be completed during the exam.

4. (**10 points.**) Question on eigenvalues and eigenbasis. To be completed during the exam.