Homework No. 07 (Fall 2020)

PHYS 520A: ELECTROMAGNETIC THEORY I

Department of Physics, Southern Illinois University–Carbondale

Due date: Friday, 2020 Oct 16, 11.00am

0. Problem 2 is to be submitted for assessment. Problem 1 is useful for conceptual understanding.

1. (**20 points.**) Recollect Legendre polynomials

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}. \tag{1}$$

In particular

$$P_0(x) = 1, \tag{2a}$$
$$P_1(x) = x, \tag{2b}$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \tag{2c}$$

Consider a charged spherical shell of radius a consisting of a charge distribution in the polar angle alone,

$$\rho(\mathbf{r}') = \sigma(\theta')\,\delta(r'-a). \tag{3}$$

The electric potential *on the z-axis*, $\theta = 0$ and $\phi = 0$, is then given by

$$\begin{aligned}\phi(r,0,0) &= \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{2\pi a^2}{4\pi\varepsilon_0}\int_0^\pi \sin\theta' d\theta' \frac{\sigma(\theta')}{\sqrt{r^2+a^2-2ar\cos\theta'}},\end{aligned} \tag{4}$$

after evaluating the r' and ϕ' integral.

(a) Consider a uniform charge distribution on the shell,

$$\sigma(\theta) = \frac{Q}{4\pi a^2}\,P_0(\cos\theta). \tag{5}$$

Evaluate the integral in Eq. (4) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_>}, \tag{6}$$

where $r_< = \mathrm{Min}(a,r)$ and $r_> = \mathrm{Max}(a,r)$.

(b) Next, consider a (pure dipole, 2×1-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_1(\cos\theta). \tag{7}$$

Evaluate the integral in Eq. (4) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{3} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right). \tag{8}$$

(c) Next, consider a (pure quadrapole, 2×2-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_2(\cos\theta). \tag{9}$$

Evaluate the integral in Eq. (4) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{5} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right)^2. \tag{10}$$

(d) For a (pure $2l$-pole) charge distribution

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_l(\cos\theta) \tag{11}$$

the integral in Eq. (4) leads to

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{(2l+1)} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right)^l. \tag{12}$$

2. (**20 points.**) The surface charge density on the surface of a charged sphere is given by

$$\sigma(\theta) = \frac{Q}{4\pi a^2} \cos^3\theta, \tag{13}$$

where θ is the polar angle in spherical coordinates. Express this charge distribution in terms of the Legendre polynomials. Recall,

$$P_0(\cos\theta) = 1, \tag{14a}$$
$$P_1(\cos\theta) = \cos\theta, \tag{14b}$$
$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}, \tag{14c}$$
$$P_3(\cos\theta) = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta. \tag{14d}$$

Determine the electric potential *on the z-axis* due to this charge distribution.