Homework No. 04 (Fall 2020)<br>PHYS 520A: ELECTROMAGNETIC THEORY I<br>Department of Physics, Southern Illinois University-Carbondale<br>Due date: Monday, 2020 Sep 21, 11.00am

1. (20 points.) An electric dipole moment $\mathbf{p}=q \mathbf{a}$ consists of two equal and opposite charges $q$ separated by a distance $\mathbf{a}$. A point electric dipole is an idealized limit of $a \rightarrow 0$, $q \rightarrow \infty$, keeping $p=a q$ fixed. The electric field of a point electric dipole moment is given by

$$
\begin{equation*}
\mathbf{E}(\mathbf{r})=\frac{1}{4 \pi \varepsilon_{0}} \frac{1}{r^{3}}[3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}}-\mathbf{p}] . \tag{1}
\end{equation*}
$$

The magnetic field $\mathbf{B}=0$ everywhere.
(a) Starting from the equation for conservation of electromagnetic linear momentum we have

$$
\begin{equation*}
\frac{\partial \mathbf{G}}{\partial t}+\boldsymbol{\nabla} \cdot \mathbf{T}+\mathbf{f}=0 \tag{2}
\end{equation*}
$$

Show that $\mathbf{G}=0$. Thus, infer

$$
\begin{equation*}
\mathbf{f} \cdot \hat{\mathbf{r}}=-(\boldsymbol{\nabla} \cdot \mathbf{T}) \cdot \hat{\mathbf{r}} . \tag{3}
\end{equation*}
$$

(b) Consider a spherical volume $V$ of radius $r$ with the point dipole at the center. Let us choose $\mathbf{p}=p \hat{\mathbf{z}}$. Note that $\mathbf{f}=\rho \mathbf{E}=-(\mathbf{p} \cdot \boldsymbol{\nabla}) \mathbf{E}$. To determine the electromagnetic stress (or the pressure, force per unit area,) on the sphere due to electrostatic repulsion between the constituent charges we define $F=\int_{V} d^{3} r(\mathbf{f} \cdot \hat{\mathbf{r}})$, which is the total sum of radial component of forces exerted on all the charges inside volume $V$ by the electric and magnetic fields. The interpretation of this quantity as the force contributing to the pressure on the charge is brought out after we use divergence theorem to learn

$$
\begin{equation*}
F=-\oint_{V} d \mathbf{a} \cdot \mathbf{T} \cdot \hat{\mathbf{r}}=-\oint_{V} d a \hat{\mathbf{r}} \cdot \mathbf{T} \cdot \hat{\mathbf{r}} \tag{4}
\end{equation*}
$$

where we used $d \mathbf{a}=d a \hat{\mathbf{r}}$. That is, $F$ is the radial force on the charge due to the flux of electromagnetic momentum across the surface enclosing volume $V$.
(c) Evaluate

$$
\begin{equation*}
\hat{\mathbf{r}} \cdot \mathbf{T} \cdot \hat{\mathbf{r}}=\frac{1}{8 \pi} \frac{1}{4 \pi \varepsilon_{0}} \frac{p^{2}}{r^{6}}\left[1-5(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})^{2}\right] \tag{5}
\end{equation*}
$$

(d) Thus, calculate the radial force $F$ on the surface of the sphere to be

$$
\begin{equation*}
F=\frac{1}{3} \frac{1}{4 \pi \varepsilon_{0}} \frac{p^{2}}{r^{4}} . \tag{6}
\end{equation*}
$$

Then, calculate the electromagnetic stress, $F /$ area, on the surface of the sphere.
2. (20 points.) Summarize Sec. III A of the article by Timothy H. Boyer titled 'Illustrations of Maxwell's term and the four conservation laws of electromagnetism' in American Journal of Physics 87 (2019) 729. Interpret the results and answer whether the energy stored in the electromagnetic fields between the plates is increasing or decreasing with time. Verify if your answer is consistent with the direction of the flux of energy.

