Midterm Exam No. 03 (Fall 2020)

PHYS 520A: ELECTROMAGNETIC THEORY I

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1. (**20 points.**) Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization transverse to the direction $\hat{\mathbf{z}}$ normal to the surface of slab, described by
$$\mathbf{P}(\mathbf{r}) = \sigma\,\hat{\mathbf{x}}\,\theta(-z), \tag{1}$$
where σ is the polarization per unit volume of the slab. Determine the effective charge density by evaluating
$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P}. \tag{2}$$

2. (**20 points.**) Consider a uniformly polarized disc of radius a that has electric polarization in the radial direction, described by
$$\mathbf{P}(\mathbf{r}) = \sigma\boldsymbol{\rho}\,\theta(a-\rho)\delta(z), \tag{3}$$
where σ is a constant and has the dimensions of charge per unit area and $\boldsymbol{\rho}$ is the radial unit vector in cylindrical polar coordinates.

   (a) Determine the effective charge density by evaluating
   $$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} \tag{4}$$
   and show that
   $$\rho_{\text{eff}}(\mathbf{r}) = -2\sigma\theta(a-\rho)\delta(z) + \sigma a\delta(\rho-a)\delta(z). \tag{5}$$
   Interpret the effective charge density. Find the total charge on the disc using $Q_{\text{en}} = \int d^3r\,\rho_{\text{eff}}(\mathbf{r})$.

   (b) Rewrite the effective charge density in spherical polar coordinates,
   $$\rho_{\text{eff}}(\mathbf{r}) = -2\sigma\frac{\delta\left(\theta-\frac{\pi}{2}\right)}{r}\theta(a-\rho) + \sigma a\frac{\delta\left(\theta-\frac{\pi}{2}\right)}{a}\delta(r-a). \tag{6}$$
   Again, find the total charge on the disc using $Q_{\text{en}} = \int d^3r\,\rho_{\text{eff}}(\mathbf{r})$.

   (c) Recall that the electric potential due to charged ring of radius a and total charge Q of charge density
   $$\rho(\mathbf{r}') = \frac{Q}{2\pi a}\frac{\delta\left(\theta'-\frac{\pi}{2}\right)}{r'}\delta(r'-a) \tag{7}$$

is given by

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}}\frac{(2n)!}{(n!)^2}\left(\frac{a}{r}\right)^{2n}P_{2n}(\cos\theta), \quad a<r. \tag{8}$$

Similarly, the electric potential due to a charged disc of radius a and total charge Q of charge density

$$\rho(\mathbf{r}') = \frac{Q}{\pi a^2}\frac{\delta\left(\theta'-\frac{\pi}{2}\right)}{r'}\theta(a-r'). \tag{9}$$

is given by

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\sum_{n=0}^{\infty}\frac{1}{(n+1)}\frac{(-1)^n}{2^{2n}}\frac{(2n)!}{(n!)^2}\left(\frac{a}{r}\right)^{2n}P_{2n}(\cos\theta), \quad a<r. \tag{10}$$

Using these results, which need not be derived here, express the electric potential due to the uniformly polarized disc in the form

$$\phi(r,\theta) = \frac{(\pi a^2)}{4\pi\varepsilon_0}\frac{\sigma}{r}\sum_{n=0}^{\infty}\alpha_n\frac{(-1)^n}{2^{2n}}\frac{(2n)!}{(n!)^2}\left(\frac{a}{r}\right)^{2n}P_{2n}(\cos\theta), \quad a<r, \tag{11}$$

and determine α_n.

3. (**20 points.**) A simple model of a metal describes the electrons in it using Newton's law,

$$m\frac{d^2\mathbf{x}}{dt^2} + m\gamma\frac{d\mathbf{x}}{dt} + m\omega_0^2\mathbf{x} = e\mathbf{E}. \tag{12}$$

Here the first term involves the acceleration of electron, ω_0-term binds the electron to the atoms, while γ-term damps the motion.

Conductor: Conductivity in typical metals is dominated by the damping term, thus

$$m\gamma\mathbf{v} = e\mathbf{E}. \tag{13}$$

The current density $\mathbf{j}$ for (constant) density n_f of conduction electrons is

$$\mathbf{j} = n_f e\mathbf{v}. \tag{14}$$

Using Eqs. (13) and (14) in conjunction we have Ohm's law

$$\mathbf{j} = \frac{n_f e^2}{m\gamma}\mathbf{E} = \sigma\mathbf{E}, \tag{15}$$

where σ is the static conductivity.

Superconductor: In 1935 Fritz London and Heinz London proposed that the current density $\mathbf{j}_s$ in a superconductor is described by the acceleration term in Eq. (12). That is,

$$m\frac{d\mathbf{v}}{dt} = e\mathbf{E}, \tag{16}$$

which together with Eq. (14) leads to London "acceleration equation"

$$\frac{d\mathbf{j}_s}{dt} = \frac{n_f e^2}{m}\mathbf{E}. \tag{17}$$

As a consequence steady currents are possible solutions when $\mathbf{E} = 0$. The insight of the London brothers led them to further propose, in addition, that the current density in a superconductor satisfies

$$\boldsymbol{\nabla} \times \left(\mathbf{j}_s + \frac{n_f e^2}{m}\mathbf{A}\right) = 0. \tag{18}$$

Thus, up to a freedom in the choice of gauge χ, we have the London equation

$$\mu_0 \, \mathbf{j}_s = -\frac{1}{\lambda_L^2}\Big(\mathbf{A} + \boldsymbol{\nabla}\chi\Big), \tag{19}$$

where λ_L defined using

$$\frac{n_f e^2}{m} = \frac{1}{\lambda_L^2}\frac{1}{\mu_0} \tag{20}$$

is the London penetration depth which is a measure of the distance magnetic field penetrates into the surface of a superconductor. The London equation replaces Ohm's law for a superconductor. Note that the London equation is consistent with the "acceleration equation" using the gauge freedom

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi, \tag{21a}$$

$$\phi' = \phi - \frac{\partial \chi}{\partial t}. \tag{21b}$$

(a) Using London's equation show that a superconductor is characterized by the equations

$$\mu_0 \frac{\partial \mathbf{j}_s}{\partial t} = \frac{1}{\lambda_L^2}\mathbf{E}, \tag{22}$$

$$\mu_0 \boldsymbol{\nabla} \times \mathbf{j}_s = -\frac{1}{\lambda_L^2}\mathbf{B}. \tag{23}$$

(b) Show that the magnetic field satisfies the equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B}. \tag{24}$$

For the static case, $\partial \mathbf{B}/\partial t = 0$, show that

$$\nabla^2 \mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B}, \tag{25}$$

which implies the Meissner effect, that a uniform magnetic field cannot exist inside a superconductor. In this static limit, and presuming planar geometry, it implies

$$\mathbf{B} = \mathbf{B}_0 \, e^{-\frac{x}{\lambda_L}}, \tag{26}$$

where the interpretation of λ_L as a penetration depth is apparent. Using Eq. (20) calculate the penetration depth for $n_f \sim 6 \times 10^{28}\,/\mathrm{m}^3$ (electron number density for gold) and show that it is of the order of tens of nanometers.