Midterm Exam No. 01 (Fall 2020)

PHYS 520A: ELECTROMAGNETIC THEORY I

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1. (**20 points.**) Evaluate
$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right), \tag{1}$$
where $\mathbf{r}$ is the coordinate vector. Deduce that Eq. (1) can not be zero everywhere in space. Express Eq. (1) in terms of δ-functions.

2. (**20 points.**) Evaluate
$$\boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho} \right). \tag{2}$$
Here (ρ, ϕ, z) are cylindrical coordinate variables. Deduce that Eq. (2) can not be zero everywhere in space. Express Eq. (2) in terms of δ-functions.

3. (**20 points.**) Given that a vector field $\mathbf{B}$ can be expressed as a curl. That is,
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{3}$$
Given $\mathbf{B} = B\,\hat{\mathbf{z}}$, where B is uniform in space. Find $\mathbf{A}$. Is your answer unique? If not, find another solution.

4. (**20 points.**) Using Gauss's law determine the charge distribution that creates a uniform radial static electric field
$$\mathbf{E} = E_0\,\hat{\mathbf{r}} \tag{4}$$
everywhere in space.

5. (**20 points.**) Let us consider the static configuration of a point electric charge q_e at a fixed position $\mathbf{r}_e$ and a point magnetic charge q_m at a fixed position $\mathbf{r}_m$. Let $\mathbf{r}_e - \mathbf{r}_m = \mathbf{a}$. For convenience we could choose the magnetic charge at the origin and the electric charge on the z axis.

   (a) Using Gauss's law show that the electric field for a (static) point electric charge is given by
$$\mathbf{E} = \frac{q_e}{4\pi\varepsilon_0} \frac{(\mathbf{r} - \mathbf{r}_e)}{|\mathbf{r} - \mathbf{r}_e|^3} = -\boldsymbol{\nabla}\phi_e, \qquad \phi_e = \frac{q_e}{4\pi\varepsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_e|}. \tag{5}$$
   Similarly, show that the magnetic field for a point (static) magnetic charge is
$$\mathbf{H} = \frac{q_m}{4\pi\mu_0} \frac{(\mathbf{r} - \mathbf{r}_m)}{|\mathbf{r} - \mathbf{r}_m|^3} = -\boldsymbol{\nabla}\phi_m, \qquad \phi_m = \frac{q_m}{4\pi\mu_0} \frac{1}{|\mathbf{r} - \mathbf{r}_m|}. \tag{6}$$

(b) Show that the electromagnetic momentum density

$$\mathbf{G} = \mathbf{D} \times \mathbf{B} \tag{7}$$

for this configuration is

$$\mathbf{G} = \varepsilon_0 \mu_0 (\boldsymbol{\nabla} \phi_e) \times (\boldsymbol{\nabla} \phi_m). \tag{8}$$

Show that $\boldsymbol{\nabla} \cdot \mathbf{G} = 0$. What is the interpretation? Thus, infer that $\mathbf{G}$ can be expressed as a curl.

(c) Show that the angular momentum density

$$\boldsymbol{l} = \mathbf{r} \times \mathbf{G} \tag{9}$$

for this configuration is

$$\boldsymbol{l} = (\mathbf{r} \cdot \mathbf{B})\mathbf{D} - (\mathbf{r} \cdot \mathbf{D})\mathbf{B}. \tag{10}$$

(d) The angular momentum is

$$\mathbf{L} = \int d^3 r \, \boldsymbol{l}, \tag{11}$$

where the integration is over all space. Show that the angular momentum for this configuration to be

$$\mathbf{L} = \frac{q_e q_m}{2\pi} \frac{1}{4\pi} \int d^3 r \, \frac{(\mathbf{r} - \mathbf{r}_e)}{|\mathbf{r} - \mathbf{r}_e|^3} \frac{1}{|\mathbf{r} - \mathbf{r}_m|}. \tag{12}$$

Hint: Show that the two terms in Eq. (10) when integrated over all space can be expressed in the form,

$$\int d^3 r \, (\mathbf{r} \cdot \mathbf{B})\mathbf{D} = \varepsilon_0 \mu_0 \int d^3 r \Big[3\mathbf{E} + (\mathbf{r} \cdot \boldsymbol{\nabla} \mathbf{E}) \Big] \phi_m, \tag{13a}$$

$$\int d^3 r \, (\mathbf{r} \cdot \mathbf{D})\mathbf{B} = \varepsilon_0 \mu_0 \int d^3 r \Big[\mathbf{E} + (\mathbf{r} \cdot \boldsymbol{\nabla} \mathbf{E}) \Big] \phi_m. \tag{13b}$$

Subtraction of these terms leads to the expression for the angular momentum in Eq. (12).

(e) Evaluate the angular momentum in Eq. (12) to be

$$\mathbf{L} = -\frac{q_e q_m}{4\pi} \, \hat{\mathbf{a}}. \tag{14}$$

The implication is that the static configuration of an electric charge and a magnetic monopole will have an angular momentum. Remarkably this angular momentum is independent of the magnitude of the distance between the monopole charges.
Hint: Recall that the electric field due to charge distribution $\rho(\mathbf{r}')$ at point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3 r' \, \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}'). \tag{15}$$

Thus, using analogy, the integrals leading to the angular momentum in Eq. (12) can be performed by evaluating the electric field due to a charge density that is inversely proportional to distance. To this end use Gauss's law to show that the electric field at the point $\mathbf{r}_e$ due to a charge density

$$\rho(\mathbf{r}) = \frac{\sigma}{|\mathbf{r} - \mathbf{r}_m|} \tag{16}$$

is

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\,\hat{\mathbf{a}}, \tag{17}$$

where $\mathbf{a} = \mathbf{r}_e - \mathbf{r}_m$.

(f) The classical configuration under consideration is upgraded to have the features of a quantum system by imposing the Bohr quantization condition

$$L = n\hbar, \qquad n = 0, 1, 2, 3, \ldots. \tag{18}$$

In this manner derive the charge quantization condition of Dirac,

$$\frac{q_e q_m}{4\pi} = n\hbar. \tag{19}$$