Homework No. 08 (Fall 2020)

PHYS 500A: MATHEMATICAL METHODS

Department of Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2020 Nov 5, 9.30am

1. (**20 points.**) Verify that
$$\frac{d}{dz}|z| = \theta(z) - \theta(-z), \tag{1}$$
where $\theta(z) = 1$, if $z > 0$, and 0, if $z < 0$. Further, verify that
$$\frac{d^2}{dz^2}|z| = 2\,\delta(z). \tag{2}$$
Also, argue that, for a well defined function $f(z)$, the replacement
$$f(z)\delta(z) = f(0)\delta(z) \tag{3}$$
is justified. Using Eq. (1), Eq. (2), and Eq. (3), verify (by substituting the solution into the differential equation) that
$$g(z) = \frac{1}{2k}\,e^{-k|z|} \tag{4}$$
is a particular solution of the differential equation
$$-\left(\frac{d^2}{dz^2} - k^2\right) g(z) = \delta(z). \tag{5}$$

2. (**20 points.**) Show that
$$\bar{\delta}(x) = -x\frac{d}{dx}\delta(x) \tag{6}$$
is also a model for the δ-function by showing that
$$\int_{-\infty}^{\infty} dx\,\bar{\delta}(x) f(x) = f(0). \tag{7}$$
Hint: Integrate by parts.

3. (**60 points.**) A forced harmonic oscillator is described by the differential equation
$$-\left(\frac{d^2}{dt^2} + \omega^2\right) x(t) = F(t), \tag{8}$$

where ω is the angular frequency of the oscillator and $F(t)$ is the forcing function. The corresponding Green's function satisfies

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G(t,t')=\delta(t-t'). \tag{9}$$

The continuity conditions satisfied by the Green function are

$$\frac{d}{dt}G(t,t')\Big|_{t=t'-\delta}^{t=t'+\delta}=-1 \tag{10}$$

and

$$G(t,t')\Big|_{t=t'-\delta}^{t=t'+\delta}=0. \tag{11}$$

(a) Verify that a particular solution,

$$G_R(t-t')=-\frac{1}{\omega}\theta(t-t')\sin\omega(t-t'), \tag{12}$$

which is referred to as the retarded Green's function, satisfies the Green function differential equation and the continuity conditions.
Hint: Use problem 2 and $\lim_{x\to\infty}\sin x/x=0$.

(b) Verify that another particular solution,

$$G_A(t-t')=\frac{1}{\omega}\theta(t'-t)\sin\omega(t-t'), \tag{13}$$

which is referred to as the advanced Green's function, satisfies the Green function differential equation and the continuity conditions.
Hint: Use problem 2 and $\lim_{x\to\infty}\sin x/x=0$.

(c) Show that the difference of the two particular solutions above,

$$G_R(t-t')-G_A(t-t'), \tag{14}$$

satisfies the homogeneous differential equations

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G_0(t,t')=0. \tag{15}$$