Homework No. 05 (Fall 2020)

PHYS 500A: MATHEMATICAL METHODS

Department of Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2020 Oct 1, 9.30am

1. (**20 points.**) Check if the function

$$f(z) = e^z + e^{iz} \tag{1}$$

satisfies the Cauchy-Riemann conditions. If $f(z)$ is analytic for all z, then report the derivative as a function of z. Otherwise, determine the points, or regions, in the z plane where the function is not analytic.

2. (**20 points.**) Check if the function

$$f(z) = \frac{1}{z} \tag{2}$$

satisfies the Cauchy-Riemann conditions.

(a) Verify that the Cauchy-Riemann conditions for this case are not well defined at $z = 0$, but are fine for $z \neq 0$.

(b) Verify that

$$\frac{df}{dz} = -\frac{1}{z^2}, \qquad z \neq 0. \tag{3}$$

(c) Determine the limiting value of the derivative as you approach $z = 0$ along the positive real line, and, then, when you approach along the negative real line. Repeat the analysis along the imaginary line. Repeat the analysis along the line $x = y$. Are these limits identical?

(d) If these limits are not identical conclude that the derivative is not isotropic at $z = 0$. Then, the function is not analytic at $z = 0$.

3. (**60 points.**) Let

$$f(z) = z^3, \tag{4}$$

so that

$$u(x, y) + iv(x, y) = r^3(\cos 3\theta + \sin 3\theta). \tag{5}$$

(a) Verify that this function satisfies the Cauchy-Riemann conditions.

(b) Show that u and v are harmonic functions. That is, they satisfy the Laplacian. Further, show that

$$(\boldsymbol{\nabla} u) \cdot (\boldsymbol{\nabla} v) = 0. \tag{6}$$

Thus, the curves represented by u and v are orthogonal at every point.

(c) Since u is a harmonic function it represents equipotential curves. Plot the equipotentials

$$r = \left[\frac{u}{\cos 3\theta}\right]^{\frac{1}{3}} \tag{7}$$

for $u = -10, -1, -0.1, 0, 0.1, 1, 10$. In Mathematica this can be achieved using the command

```
PolarPlot[{r[-10],...,r[10]},{th,0,2 Pi}],
```

where `r[u]` a function of u and `th` needs to be defined ahead.

(d) Determine the electric field associated to these equipotentials using

$$\mathbf{E} = -\boldsymbol{\nabla} u. \tag{8}$$

This is easily achieved using

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \tag{9}$$

and similarly for derivatives with respect to y. Recall

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}. \tag{10}$$

Show that

$$\mathbf{E} = -\hat{\mathbf{i}}\, 3r^2\cos 2\theta + \hat{\mathbf{j}}\, 3r^2 \sin 2\theta. \tag{11}$$

(e) The curves representing the field lines are obtained by requiring the tangent lines for these curves to have the same slope as the electric field,

$$\frac{dx}{E_x} = \frac{dy}{E_y}. \tag{12}$$

Rewrite this equation as

$$E_y dx - E_x dy = 0. \tag{13}$$

Comparing this equation with

$$\frac{\partial s}{\partial x}\, dx + \frac{\partial s}{\partial y}\, dy = 0 \tag{14}$$

identify the equations satisfied by the curves $s(x, y)$, representing the field lines associated to the equipotentials u, as

$$\frac{\partial s}{\partial x} = 6xy, \qquad \frac{\partial s}{\partial y} = 3(x^2 - y^2). \tag{15}$$

Solve these equations to determine the equations for the field lines to be

$$s(x, y) = 3x^2 y - y^3 = r^3 \sin 3\theta \tag{16}$$

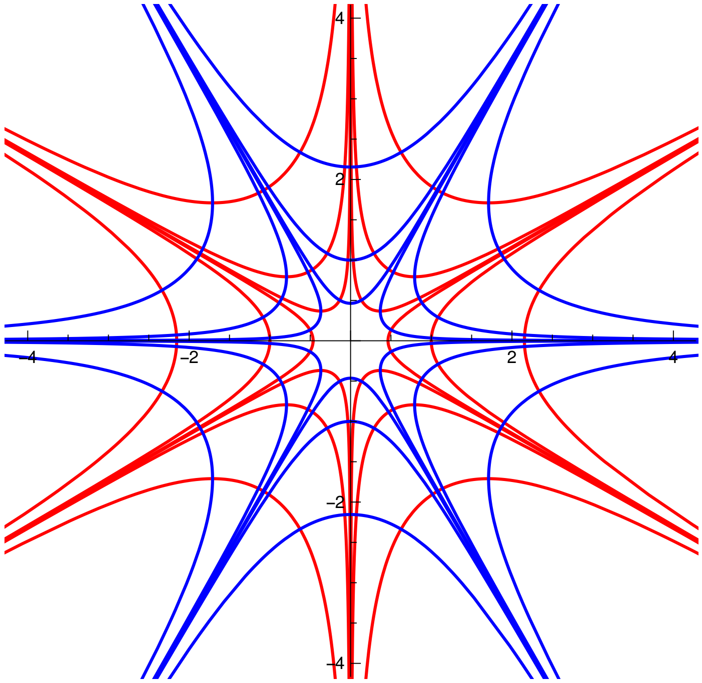

Figure 1: Equipotentials and field lines represented by the analytic function $f(z) = z^3$.

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up to a constant. The field lines s are indeed v. Plot the field lines

$$r = \left[\frac{v}{\sin 3\theta}\right]^{\frac{1}{3}} \tag{17}$$

for $v = -10, -1, -0.1, 0, 0.1, 1, 10$.

(f) Plot the equipotentials in red and field lines in blue in the same plot. Here is a simple code for it in Mathematica

```
n = 3;
f[u_] = (u/Cos[n t])^(1/n);
g[u_] = (u/Sin[n t])^(1/n);
PolarPlot[
 {f[-10], f[-1], f[-0.1], f[0], f[0.1], f[1], f[10],
  g[-10], g[-1], g[-0.1], g[0], g[0.1], g[1], g[10]},
 {t, -Pi, Pi},
 PlotStyle -> {Red, Red, Red, Red, Red, Red, Red,
               Blue, Blue, Blue, Blue, Blue, Blue, Blue},
 PlotRange -> {-4, 4}]
```

which generates the plots in Fig. 1.