

Homework No. 02 (Fall 2020)

PHYS 500A: MATHEMATICAL METHODS

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Due date: Tuesday, 2020 Sep 1, 9.30am

1. **(Example.)** Let \mathbf{r} represent a position vector in three dimensional space. Let x^i be the components of the position vector in rectangular coordinates, which can be interpreted as surfaces of constant x^i . Let us coordinatize the space using the planes, labeled using β ,

$$y = mx + \beta \tag{1}$$

where m is fixed, instead of planes with constant y . The other two sets of planes of constant x and constant z are the same. See Fig. 1. Let u^i be the components of the position vector in this new coordinatization of space. In particular, we have

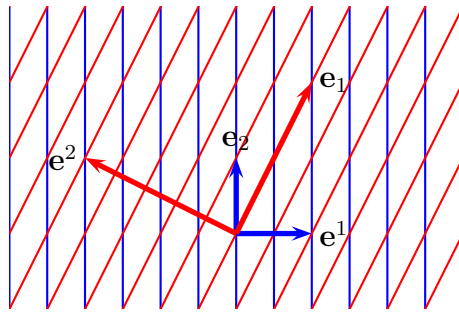


Figure 1: Basis vectors \mathbf{e}_i and reciprocal basis vectors \mathbf{e}^i .

$$x^1 = x = \alpha, \qquad u^1 = \alpha = x, \tag{2a}$$

$$x^2 = y = mx + \beta, \qquad u^2 = \beta = y - mx, \tag{2b}$$

$$x^3 = z = \gamma, \qquad u^3 = \gamma = z. \tag{2c}$$

The basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ in rectangular coordinate system will be represented as $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, if necessary.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{3}$$

Show that

$$\mathbf{e}_1 = \hat{\mathbf{i}} + m\hat{\mathbf{j}}, \quad \mathbf{e}_2 = \hat{\mathbf{j}}, \quad \mathbf{e}_3 = \hat{\mathbf{k}}. \tag{4}$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \nabla u^i. \quad (5)$$

Show that

$$\mathbf{e}^1 = \hat{\mathbf{i}}, \quad \mathbf{e}^2 = -m\hat{\mathbf{i}} + \hat{\mathbf{j}}, \quad \mathbf{e}^3 = \hat{\mathbf{k}}. \quad (6)$$

Verify the relations

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_2 \times \mathbf{e}_3) \cdot \mathbf{e}_1}, \quad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{(\mathbf{e}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_2}, \quad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3}. \quad (7)$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i. \quad (8)$$

That is,

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = 1, \quad \mathbf{e}^1 \cdot \mathbf{e}_2 = 0, \quad \mathbf{e}^1 \cdot \mathbf{e}_3 = 0, \quad (9a)$$

$$\mathbf{e}^2 \cdot \mathbf{e}_1 = 0, \quad \mathbf{e}^2 \cdot \mathbf{e}_2 = 1, \quad \mathbf{e}^2 \cdot \mathbf{e}_3 = 0, \quad (9b)$$

$$\mathbf{e}^3 \cdot \mathbf{e}_1 = 0, \quad \mathbf{e}^3 \cdot \mathbf{e}_2 = 0, \quad \mathbf{e}^3 \cdot \mathbf{e}_3 = 1. \quad (9c)$$

(d) Metric tensor: The metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \quad (10)$$

Evaluate all the components of g_{ij} . That is,

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1 + m^2, \quad g_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2 = m, \quad g_{13} = \mathbf{e}_1 \cdot \mathbf{e}_3 = 0, \quad (11a)$$

$$g_{21} = \mathbf{e}_2 \cdot \mathbf{e}_1 = m, \quad g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1, \quad g_{23} = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0, \quad (11b)$$

$$g_{31} = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \quad g_{32} = \mathbf{e}_3 \cdot \mathbf{e}_2 = 0, \quad g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1. \quad (11c)$$

Similarly evaluate the components of

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j. \quad (12)$$

That is,

$$g^{11} = \mathbf{e}^1 \cdot \mathbf{e}^1 = 1, \quad g^{12} = \mathbf{e}^1 \cdot \mathbf{e}^2 = -m, \quad g^{13} = \mathbf{e}^1 \cdot \mathbf{e}^3 = 0, \quad (13a)$$

$$g^{21} = \mathbf{e}^2 \cdot \mathbf{e}^1 = -m, \quad g^{22} = \mathbf{e}^2 \cdot \mathbf{e}^2 = 1 + m^2, \quad g^{23} = \mathbf{e}^2 \cdot \mathbf{e}^3 = 0, \quad (13b)$$

$$g^{31} = \mathbf{e}^3 \cdot \mathbf{e}^1 = 0, \quad g^{32} = \mathbf{e}^3 \cdot \mathbf{e}^2 = 0, \quad g^{33} = \mathbf{e}^3 \cdot \mathbf{e}^3 = 1. \quad (13c)$$

Verify that $g^{ij}g_{jk} = \delta_k^i$.

(e) Completeness relation: Verify the completeness relation

$$\mathbf{e}^i \mathbf{e}_i = \mathbf{1} \quad (14)$$

by evaluating

$$\mathbf{e}^1 \mathbf{e}_1 + \mathbf{e}^2 \mathbf{e}_2 + \mathbf{e}^3 \mathbf{e}_3. \quad (15)$$

(f) Given a vector

$$\mathbf{A} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}} \quad (16)$$

in rectangular coordinates, find the components of the vector \mathbf{A} in the basis of \mathbf{e}_i . That is, find the components A^i in

$$\mathbf{A} = A^1 \mathbf{e}_1 + A^2 \mathbf{e}_2 + A^3 \mathbf{e}_3. \quad (17)$$