Midterm Exam No. 03 (Fall 2020)

PHYS 500A: MATHEMATICAL METHODS

Department of Physics, Southern Illinois University–Carbondale
Due date: Tuesday, 2020 Nov 10, 9.30am

0. **Note:** This exam is being used to cover content. Thus, you are allowed to take complete help from all resources, including me. Discussions between peers taking this exam, including looking into each others work, is encouraged. Individual hand-written, or individually typographed, submissions are expected. Problem 2 conceptually follows Problem 1, however, when seen as an exercise, Problem 2 is simpler. As a resource material on Method of Images following references are recommended:

   (a) YouTube Playlist for PHYS-320 at SIUC this Fall: SIUC-PHYS-320 (2020 Fall). Lectures starting from 20201019 are relevant.

   (b) Chapters 14 and 17 in the book titled *Classical Electrodynamics*, by J. Schwinger, L. L. DeRaad, Jr., K. A. Milton, and Wu-yang Tsai.

1. (**50 points.**) Let the space be filled with two dielectric materials, with a discontinuity at $z = 0$, such that

$$\varepsilon(\mathbf{r}) = \varepsilon_2\theta(-z) + \varepsilon_1\theta(z), \tag{1}$$

where

$$\varepsilon_2 < \varepsilon_1. \tag{2}$$

In addition there is a point charge at q at

$$\mathbf{r}' = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + a\,\hat{\mathbf{z}}. \tag{3}$$

In the following we shall determine the electric potential and electric field everywhere for this configuration.

   (a) Starting from the Maxwell equations (in vacuum) the electric potential for a single point charge at $\mathbf{r}'$ is

$$-\varepsilon_0\nabla^2\phi(\mathbf{r}) = q\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{4}$$

   Construct the corresponding Green's function to satisfy

$$-\varepsilon_0\nabla^2 G_0(\mathbf{r}-\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}'), \tag{5}$$

   which is obtained by replacing the charge density with that of a point charge of unit magnitude that is achieved by simply choosing $q = 1$ in this case. The electric potential is in general given in terms of the Green's function by the superposition principle

$$\phi(\mathbf{r}) = \int d^3r' G_0(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}'), \tag{6}$$

which for a point charge in vacuum simply reads

$$\phi(\mathbf{r}) = q\, G_0(\mathbf{r},\mathbf{r}'), \qquad G_0(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}, \tag{7}$$

Observe that the configuration under consideration, a point charge in a planar dielectric region, has translation symmetry in x and y directions. Thus, we use Fourier transformation in these coordinates to write

$$G_0(\mathbf{r},\mathbf{r}') = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp} g_0(z,z';k_\perp), \tag{8}$$

where $g_0(z,z';k_\perp)$ is the Fourier transform of $G_0(\mathbf{r},\mathbf{r}')$ in the x and y coordinates. Here the subsript $\perp$ is the projection in the plane perpendicular to $\hat{\mathbf{z}}$. Show that the reduced Green's function $g_0(z,z';k_\perp)$ satisfies the differential equation

$$-\left(\frac{d^2}{dz^2} - k_\perp^2\right)\varepsilon_0 g_0(z,z';k_\perp) = \delta(z-z'). \tag{9}$$

Show that

$$g_0(z,z';k_\perp) = \frac{1}{\varepsilon_0}\frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}. \tag{10}$$

Thus, find the identity

$$\begin{aligned}\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{4\pi}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ &= \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp} \frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}.\end{aligned} \tag{11}$$

(b) Starting from the macroscopic Maxwell equations the electric potential for a single point charge at $\mathbf{r}'$ in the presence of a dielectric material is

$$-\boldsymbol{\nabla}\cdot\Big[\varepsilon(\mathbf{r})\boldsymbol{\nabla}\Big]\phi(\mathbf{r}) = q\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{12}$$

Construct the corresponding Green's function to satisfy

$$-\boldsymbol{\nabla}\cdot\Big[\varepsilon(\mathbf{r})\boldsymbol{\nabla}\Big]G(\mathbf{r},\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{13}$$

Show that the corresponding reduced Green's function $g(z,z';k_\perp)$ satisfies the differential equation

$$\left[-\frac{\partial}{\partial z}\varepsilon(z)\frac{\partial}{\partial z} + \varepsilon(z)k_\perp^2\right] g(z,z';k_\perp) = \delta(z-z'), \tag{14}$$

where

$$\varepsilon(z) = \begin{cases}\varepsilon_2, & z<0,\\ \varepsilon_1, & 0<z.\end{cases} \tag{15}$$

Show that

$$g(z,z') = \begin{cases} \frac{1}{\varepsilon_2}\frac{1}{2k_\perp}e^{-k_\perp|z-z'|} + \frac{1}{\varepsilon_2}\frac{1}{2k_\perp}\left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right)e^{-k_\perp|z|}e^{-k_\perp|z'|}, & z'<0, \\ \frac{1}{\varepsilon_1}\frac{1}{2k_\perp}e^{-k_\perp|z-z'|} + \frac{1}{\varepsilon_1}\frac{1}{2k_\perp}\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)e^{-k_\perp|z|}e^{-k_\perp|z'|}, & 0<z'. \end{cases} \tag{16}$$

(c) Use the identity in Eq. (11) to show that

$$\phi(\mathbf{r}) = q\,G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\varepsilon_1}\frac{q}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\varepsilon_1}\frac{q_{\rm im}}{|\mathbf{r}-\mathbf{r}'_{\rm im}|}, \tag{17}$$

where

$$q_{\rm im} = -q\left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right) \tag{18}$$

and

$$\mathbf{r}'_{\rm im} = \begin{cases} \mathbf{r}', & z>0, \\ \mathbf{r}' - 2a\,\hat{\mathbf{z}}, & z<0. \end{cases} \tag{19}$$

Thus, prescribe an algorithm to determine the electric potential for planar dielectrics–a method of images for planar dielectrics.

(d) Determine the electric field to be

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_1}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{q_{\rm im}}{4\pi\varepsilon_1}\frac{\mathbf{r}-\mathbf{r}'_{\rm im}}{|\mathbf{r}-\mathbf{r}'_{\rm im}|^3}. \tag{20}$$

Draw the electric field lines for $\varepsilon_2 < \varepsilon_1$, and compare it with the electric field lines for $\varepsilon_2 > \varepsilon_1$.

(e) By evaluating the ratios

$$\frac{E_x(x,y,+\delta)}{E_x(x,y,-\delta)}, \quad \frac{E_y(x,y,+\delta)}{E_y(x,y,-\delta)}, \quad \frac{E_z(x,y,+\delta)}{E_z(x,y,-\delta)}, \tag{21}$$

determine the boundary conditions satisfied by the electric field lines. This is the Snell's law for the electric field lines. Note that the Snell's law for refraction is expressed in terms of the propagation vector of a plane wave, which is perpendicular to the electric field lines.

(f) A perfect conductor (in the static limit) is a dielectric material with a very high dielectric constant ($\varepsilon \to \infty$). Consider the extreme limit

$$\varepsilon_1 < \varepsilon_2 \to \infty \tag{22}$$

and

$$\varepsilon_2 < \varepsilon_1 \to \infty. \tag{23}$$

Examine these cases critically. Compare your results with the method of images for perfect conductors.

2. (**50 points.**) The expression for the electric potential due to a point charge placed in front of a perfectly conducting semi-infinite slab described by

$$\varepsilon(z) = \begin{cases} \infty, & z < 0, \\ \varepsilon_0, & 0 < z, \end{cases} \tag{24}$$

is given in terms of the reduced Green's function that satisfies the differential equation $(0 < \{z, z'\})$

$$-\left[\frac{\partial^2}{\partial z^2} - k^2\right] \varepsilon_0 g(z, z') = \delta(z - z') \tag{25}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$.

(a) Construct the reduced Green's function in the form

$$g(z, z') = \begin{cases} Ae^{kz} + Be^{-kz}, & 0 < z < z', \\ Ce^{kz} + De^{-kz}, & 0 < z' < z, \end{cases} \tag{26}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$g(0, z') = 0, \tag{27a}$$

$$g(a, z') = 0, \tag{27b}$$

$$g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{27c}$$

$$\varepsilon_0 \partial_z g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{27d}$$

(b) Express the solution the form

$$g(z, z') = \frac{1}{\varepsilon_0}\frac{1}{2k} e^{-k|z-z'|} - \frac{1}{\varepsilon_0}\frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{28}$$

(c) Deduce the method of images from the above solution.