Midterm Exam No. 01 (Fall 2020)

PHYS 500A: MATHEMATICAL METHODS

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1. (**20 points.**) In terms of spherical coordinates (r, θ, ϕ) it is given that

$$\frac{\partial}{\partial \phi}\hat{\boldsymbol{\phi}} = a\,\hat{\mathbf{r}} + b\,\hat{\boldsymbol{\theta}} + c\,\hat{\boldsymbol{\phi}}. \tag{1}$$

   Find the expression for the components a, b, and c, such that the above equation is an identity.

2. (**20 points.**) In terms of cylindrical coordinates (ρ, ϕ, z) we can write

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\rho + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{2}$$

$$\boldsymbol{\nabla}_\rho = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi}. \tag{3}$$

   Verify the following identities:

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 2\pi\delta^{(2)}(\boldsymbol{\rho}), \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 0, \tag{4a}$$

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = 0, \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = \hat{\mathbf{z}}\,2\pi\delta^{(2)}(\boldsymbol{\rho}). \tag{4b}$$

   Hint: Evaluate the quantities for $\rho \neq 0$. Then, use the divergence theorem or the Stoke's theorem for arbitrary volumes and surfaces in conjunction with the definition of δ-function.

3. (**20 points.**) In terms of cylindrical coordinates (ρ, ϕ, z) we can write

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{5}$$

   Given

$$\mathbf{B} = \hat{\mathbf{z}}2\ln\frac{2L}{\rho}, \tag{6}$$

   where L is a constant. Evaluate (without switching coordinate systems)

$$\boldsymbol{\nabla}\cdot\mathbf{B} \quad \text{for} \quad \rho \neq 0 \tag{7}$$

   and

$$\boldsymbol{\nabla}\times\mathbf{B} \quad \text{for} \quad \rho \neq 0. \tag{8}$$

4. (**20 points.**) In terms of cylindrical coordinates (ρ, ϕ, z) we can write

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{9}$$

Derive the expression for the Laplacian in cylindrical coordinates.

5. (**20 points.**) In terms of unit vectors

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{10a}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{10b}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}, \tag{10c}$$

the basis vectors for spherical polar coordinates are

$$\mathbf{e}_r = \hat{\mathbf{r}} \qquad \mathbf{e}^r = \hat{\mathbf{r}}, \tag{11a}$$

$$\mathbf{e}_\theta = r\hat{\boldsymbol{\theta}} \qquad \mathbf{e}^\theta = \frac{1}{r}\hat{\boldsymbol{\theta}}, \tag{11b}$$

$$\mathbf{e}_\phi = r\sin\theta\hat{\boldsymbol{\phi}} \qquad \mathbf{e}^\phi = \frac{1}{r\sin\theta}\hat{\boldsymbol{\phi}}. \tag{11c}$$

Given the expression for the Christoffel symbols

$$\Gamma^k_{ij} = \left(\frac{\partial}{\partial u^j}\mathbf{e}_i\right)\cdot\mathbf{e}^k \tag{12}$$

for the spherical coordinate system, evaluate

$$\Gamma^\phi_{\theta\phi}. \tag{13}$$