Final Exam (Fall 2020)

PHYS 500A: MATHEMATICAL METHODS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2020 Dec 8, 10.00am

1. (**20 points.**) Starting from

$$\frac{\delta(\rho-\rho')\delta(\phi-\phi')}{\rho} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^\infty k_\perp dk_\perp J_m(k_\perp\rho)J_m(k_\perp\rho') \tag{1}$$

show that

$$\frac{\delta(\rho-\rho')}{\rho} = \int_0^\infty k_\perp dk_\perp J_m(k_\perp\rho)J_m(k_\perp\rho'). \tag{2}$$

Hint: Multiply the first equation by $e^{-im'(\phi-\phi')}$ on both sides, and integrate with respect to ϕ. Use the property of δ-function on the left hand side, and

$$\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(m-m')(\phi-\phi')} = \delta_{mm'} \tag{3}$$

on the right hand side.

2. (**20 points.**) The generating function for the spherical harmonics, $Y_{lm}(\theta,\phi)$, is

$$\frac{1}{l!}\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right)^l = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\,Y_{lm}(\theta,\phi)\,\psi_{lm}, \tag{4}$$

where the left hand side is expressed in terms of

$$\mathbf{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{5}$$

$$\mathbf{a} = \frac{1}{2}(y_-^2 - y_+^2, -iy_-^2 - iy_+^2, 2y_-y_+), \tag{6}$$

and the right hand side consists of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}}\frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{7}$$

and

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}. \tag{8}$$

Show that

$$\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right) \tag{9}$$

is unchanged by the substitution: $y_+ \leftrightarrow y_-$, $\theta\to-\theta$, $\phi\to-\phi$. Thus, show that

$$Y_{lm}(\theta,\phi) = Y_{l,-m}(-\theta,-\phi). \tag{10}$$

3. (**20 points.**) Write down the explicit forms of the spherical harmonics $Y_{lm}(\theta, \phi)$ for $l = 0, 1, 2$, by completing the $l - m$ differentiations in Eq. (8). Use the result in Eq. (10) to reduce the work by about half.

4. (**20 points.**) Legendre polynomials of order l is given by (for $|t| < 1$)

$$P_l(t) = \left(\frac{d}{dt}\right)^l \frac{(t^2 - 1)^l}{2^l l!}. \tag{11}$$

   (a) Write down the explicit forms of the Legendre polynomials $P_l(t)$ for $l = 0, 1, 2, 3$, by completing the l differentiations in Eq. (11).

   (b) Show that the spherical harmonics for $m = 0$ involves the Legendre polynomials,

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi}} P_l(\cos\theta). \tag{12}$$

   (c) Using the orthonormality condition for the spherical harmonics

$$\int d\Omega\, Y^*_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'}\delta_{mm'} \tag{13}$$

   recognize the orthogonality statement for Legendre polynomials,

$$\frac{1}{2}\int_{-1}^{1} dt\, P_l(t) P_{l'}(t) = \frac{\delta_{ll'}}{2l + 1}. \tag{14}$$

   Use

$$P_0(t) = 1, \quad P_1(t) = t, \quad P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}, \tag{15}$$

   to check this explicitly for $l, l' = 0, 1, 2$.

End of Exam