Homework No. 09 (2020 Fall)

PHYS 320: ELECTRICITY AND MAGNETISM I

Due date: Monday, 2020 Oct 19, 2:00 PM, on D2L

0. Keywords: Electric polarization, dielectric function

0. Problems 1, 2, 4, and 7 are to be submitted for assessment. Rest are for practice.

1. (**Example.**) Consider a uniformly polarized slab of thickness a, that has the direction of its electric polarization (electric dipole moment density) in the direction $\hat{\mathbf{z}}$ that is normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma\,\hat{\mathbf{z}}\left[\theta(z) - \theta(z-a)\right] = \begin{cases} 0, & z < 0, \\ \sigma\,\hat{\mathbf{z}}, & 0 < z < a, \\ 0, & a < z, \end{cases} \tag{1}$$

where σ characterizes the polarization of the slab. Note that σ is dipole moment per unit volume, which has dimensions of charge per unit area.

   (a) Determine the effective charge density by evaluating

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} \tag{2}$$

   and show that

$$\rho_{\text{eff}}(\mathbf{r}) = -\sigma\delta(z) + \sigma\delta(z-a). \tag{3}$$

   Interpret the effective charge density as a surface charge density. Draw a diagram illustrating how the distribution of dipole moment density $\mathbf{P}$ leads to a surface charge density.

   (b) Find the total charge in the slab using

$$Q_{\text{en}} = \int d^3r\,\rho_{\text{eff}}(\mathbf{r}). \tag{4}$$

2. (**20 points.**) Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization transverse to the direction $\hat{\mathbf{z}}$ normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma\,\hat{\mathbf{x}}\,\theta(-z), \tag{5}$$

where σ is the polarization per unit area of the slab. Determine the effective charge density by evaluating

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P}. \tag{6}$$

3. (**Example.**) Consider a solid sphere of radius R with uniform permanent polarization

$$\mathbf{P}(\mathbf{r},t) = \mathbf{P}_0\,\theta(R-r), \tag{7}$$

where $\mathbf{P}_0$ is a uniform vector, $\theta(x)$ is the Heaviside step function, and $r^2 = x^2 + y^2 + z^2$.

   (a) Show that the effective charge density due to the polarization is

   $$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} = (\mathbf{P}_0\cdot\hat{\mathbf{r}})\,\delta(r-R). \tag{8}$$

   (b) If we choose polarization to be along the direction of $\hat{\mathbf{z}}$, that is,

   $$\mathbf{P}_0 = \sigma\,\hat{\mathbf{z}}, \tag{9}$$

   we have, using $\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = \cos\theta$,

   $$\rho_{\text{eff}}(\mathbf{r}) = \sigma\cos\theta\,\delta(r-R). \tag{10}$$

   Interpret the effective charge density as a surface charge density. Draw a diagram illustrating how the distribution of dipole moment density $\mathbf{P}$ leads to a surface charge density.

   (c) Find the enclosed charge inside an arbitrary sphere of radius r using

   $$Q_{\text{en}} = \int d^3r\,\rho_{\text{eff}}(\mathbf{r}) \tag{11}$$

   for $r < R$ and $r > R$.

4. (**20 points.**) Consider a uniformly polarized sphere of radius R described by

$$\mathbf{P}(\mathbf{r}) = \alpha\,\mathbf{r}\,\theta(R-r). \tag{12}$$

   (a) Calculate $-\boldsymbol{\nabla}\cdot\mathbf{P}$. Thus, find the effective charge density to be

   $$\rho_{\text{eff}} = -3\alpha\theta(R-r) + \alpha r\delta(r-R). \tag{13}$$

   (b) Find the enclosed charge inside a sphere of radius r using

   $$Q_{\text{en}} = \int d^3r'\,\rho_{\text{eff}}(\mathbf{r}') \tag{14}$$

   for $r < R$ and $r > R$.

5. (**20 points.**) A permanently polarized sphere of radius R is described by the polarization vector

$$\mathbf{P}(\mathbf{r}) = \alpha r^2\,\hat{\mathbf{r}}\,\theta(R-r). \tag{15}$$

Find the effective charge density by calculating $-\boldsymbol{\nabla}\cdot\mathbf{P}$. In particular, you should obtain two terms, one containing $\theta(R-r)$ that is interpreted as a volume charge density, and another containing $\delta(R-r)$ that can be interpreted as a surface charge density.

6. (**Example.**) Consider a solid right circular cylinder of radius R with axis along the z axis and of infinite length with uniform permanent polarization

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_0\, \theta(R - \rho), \tag{16}$$

where $\rho^2 = x^2 + y^2$ and $\mathbf{P}_0$ is perpendicular to the axis of the cylinder. We shall find the electric potential and the electric field outside the cylinder.

   (a) Show that the effective charge density is given by the expression

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla} \cdot \mathbf{P} = \mathbf{P}_0 \cdot \hat{\boldsymbol{\rho}}\, \delta(\rho - R). \tag{17}$$

   (b) Discuss the case when $\mathbf{P}_0$ is parallel to the axis of the cylinder. Further, qualitatively, discuss the case if, in addition, the cylinder was of finite length in the direction of z.

7. (**Example.**) Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization in the direction $\hat{\mathbf{z}}$ that is normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma\, \hat{\mathbf{z}}\, \theta(-z), \tag{18}$$

where σ is the polarization of the slab. Determine the electric field, inside and outside the slab?

8. (**20 points.**) Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization transverse to the direction $\hat{\mathbf{z}}$ that is normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma\, \hat{\mathbf{x}}\, \theta(-z), \tag{19}$$

where σ is the polarization of the slab. Determine the electric field, inside and outside the slab?