Homework No. 07 (2020 Fall)

PHYS 320: ELECTRICITY AND MAGNETISM I

Due date: Wednesday, 2020 Sep 30, 2:00 PM, on D2L

0. Problems 1, 4, 6, and 7, are to be submitted for assessment. Rest are for practice.

   Keywords: Multipole expansion, monopole moment, dipole moment, quadrupole moment, electric potential due to multipoles.

1. (**20 points.**) Consider a configuration of charges $q_1, q_2, q_3, \ldots$, at positions $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots$, and let $\mathbf{r}_0$ be an arbitrary point in space. Define the postion vector of the charges with respect to $\mathbf{r}_0$ to be

$$\mathbf{R}_i = \mathbf{r}_i - \mathbf{r}_0. \tag{1}$$

   The monopole moment, the dipole moment, and the quadrupole moment of this configuration is given by

$$Q = q_1 + q_2 + q_3 + \ldots, \tag{2a}$$

$$\mathbf{d} = q_1\mathbf{R}_1 + q_2\mathbf{R}_2 + q_3\mathbf{R}_3 + \ldots, \tag{2b}$$

$$\mathbf{q} = q_1(3\mathbf{R}_1\mathbf{R}_1 - R_1^2\mathbf{1}) + q_2(3\mathbf{R}_2\mathbf{R}_2 - R_2^2\mathbf{1}) + q_3(3\mathbf{R}_3\mathbf{R}_3 - R_3^2\mathbf{1}) + \ldots, \tag{2c}$$

   respectively. Evaluate the monopole moment, the dipole moment, and the quadrupole moment of three identical charges, each having charge q, positioned on the x axis at a, $2a$, and $3a$, respectively.

2. (**20 points.**) Evaluate the monopole moment, the dipole moment, and the quadrupole moment of countable infinite identical charges, each having charge q, positioned on the x axis at $a, a/2, a/3, \ldots$, respectively.

3. (**20 points.**) The monopole moment, the dipole moment, and the quadrupole moment, of a charge distribution $\rho(\mathbf{r})$ is given by

$$Q = \int d^3r\, \rho(\mathbf{r}), \tag{3a}$$

$$\mathbf{d} = \int d^3r\, \rho(\mathbf{r})\, \mathbf{r}, \tag{3b}$$

$$\mathbf{q} = \int d^3r\, \rho(\mathbf{r}) \Big[3\mathbf{r}\mathbf{r} - r^2\mathbf{1}\Big], \tag{3c}$$

   respectively. Consider a charge distribution consisting of a single point charge. If it is placed at the origin calculate the monopole moment, dipole moment, and quadrupole moment, of the charge distribution. Repeat the calculation if the position of the point charge is $(a, 0, 0)$.

4. (**20 points.**) Show that a configuration consisting of three charges with zero electric monopole moment and zero electric dipole moment is collinear.
   Hint: Let the three charges be q_1, q_2, and q_3, and their positions be $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, respectively. Show that we can express $(\mathbf{r}_1 - \mathbf{r}_3) = a(\mathbf{r}_1 - \mathbf{r}_2)$ and $(\mathbf{r}_2 - \mathbf{r}_3) = b(\mathbf{r}_1 - \mathbf{r}_2)$. Find a and b.

5. (**20 points.**) Two charges with charge $+q$ and $-q$ are placed at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. Find the monopole moment and the dipole moment of this configuration of three charges. Is the dipole moment independent of the choice of origin? Is the dipole moment independent of the orientation of the coordinate axis?

6. (**20 points.**) Two charges with charge $+q$ each are placed at $(a, 0, 0)$ and $(-a, 0, 0)$. A third charge with charge $-2q$ is placed at the origin. Find the monopole moment, the dipole moment, and the quadrupole moment, of this configuration of three charges.

7. (**20 points.**) A positive charge q is placed at $(a, 0, 0)$. Two negative charges of charge $-q$ each are placed at $(-a/2, a\sqrt{3}/2, 0)$ and $(-a/2, -a\sqrt{3}/2, 0)$. Find the monopole moment, dipole moment, and the quadrupole moment, of this configuration of charges.

8. (**20 points.**) Two electrons and two protons are placed at the corners of a square of length a, such that the electrons are at diagonally opposite corners. For simplicity let us choose them to be in the xy plane. Find the monopole moment, the dipole moment, and the quadrupole moment, of this configuration of four charges. Do these moments depend on the orientation of the square in the xy plane?

9. (**20 points.**) Two electrons and two protons are placed at the corners of a rectangle of length a and width b, such that the electrons are at diagonally opposite corners. For simplicity let us choose them to be in the xy plane. Find the monopole moment, the dipole moment, and the quadrupole moment, of this configuration of four charges. Do these moments depend on the orientation of the square in the xy plane?

10. (**20 points.**) We have three charges q_1, q_2, and q_3, at positions $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, respectively. If the configuration has zero electric monopole moment and zero electric dipole moment, then show that the three charges are collinear. Further, show that the electric quadrupole moment of the configuration is

$$\mathbf{q} = q_h \Big[3(\mathbf{r}_1 - \mathbf{r}_2)(\mathbf{r}_1 - \mathbf{r}_2) - (\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2) \mathbf{1} \Big]. \tag{4}$$

    where q_h is the harmonic mean of q_1 and q_2 given by

$$\frac{1}{q_h} = \frac{1}{q_1} + \frac{1}{q_2}. \tag{5}$$