Homework No. 06 (2020 Fall)

PHYS 320: ELECTRICITY AND MAGNETISM I

Due date: Monday, 2020 Sep 21, 2:00 PM, on D2L

0. Problems 1, 2, and 7, are to be submitted for assessment. Rest are for practice.

1. (**20 points.**) Two electrons and two protons are placed at the corners of a square of side a, such that the electrons are at diagonally opposite corners.

   (a) What is the electric potential at the center of square?

   (b) What is the electric potential at the midpoint of either one of the sides of the square?

   (c) How much potential energy is required to move another proton from infinity to the center of the square?

   (d) How much additional potential energy is required to move the proton from the center of the square to one of the midpoint of either one of the sides of the square?

2. (**20 points.**) (Griffiths 4th edition, Problem 2.32.) Two positive charges, q_1 and q_2 (masses m_1 and m_2) are at rest, held together by a massless string of length a. Now the string is cut, and the particles fly off in opposite directions. How fast is each one going, when they are far apart?

3. (**20 points.**) The charge density for a point charge q_a is described by

$$\rho(\mathbf{r}) = q_a \delta^{(3)}(\mathbf{r} - \mathbf{r}_a), \tag{1}$$

   where $\mathbf{r}_a$ is the position of the charge.

   (a) Evaluate the electric potential due to the point charge using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{2}$$

   (Hint: Use the δ-function property to evaluate the integrals.)

   (b) Evaluate the electric field due to the point charge by finding the gradient of the electric potential you calculated using Eq. (2),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{3}$$

   (c) Evaluate the force exerted by the charge q_a on another charge q_b, at position $\mathbf{r}_b$, using the expression for electric field you obtained using Eq. (3) in

$$\mathbf{F} = q_b \mathbf{E}(\mathbf{r}_b). \tag{4}$$

To provide a check for your calculation, the answer for the expression for the force is provided here:

$$\mathbf{F} = \frac{q_a q_b}{4\pi\varepsilon_0} \frac{\mathbf{r}_b - \mathbf{r}_a}{|\mathbf{r}_b - \mathbf{r}_a|^3}. \tag{5}$$

4. (**20 points.**) The charge density of a uniformly charged sphere of radius R with total charge Q is described by

$$\rho(\mathbf{r}) = \frac{Q}{\frac{4}{3}\pi R^3}\theta(R - r). \tag{6}$$

   (a) Evaluate the integral

$$\int d^3r' \rho(\mathbf{r}') \tag{7}$$

   over all space.

   (b) Evaluate the electric potential of the sphere inside and outside the sphere using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{8}$$

   (Hint: Choose the observation pont to be on the z axis, which allows the θ' and ϕ' integrals to be evaluated. Then, complete the r' integral.)

   (c) Evaluate the electric field due to the point charge by finding the gradient of the electric potential you calculated using Eq. (8),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{9}$$

5. (**20 points.**) The charge density for a perfectly conducting sphere of radius R with total charge Q on it is described by

$$\rho(\mathbf{r}) = \frac{Q}{4\pi R^2}\delta(r - R). \tag{10}$$

   (a) Evaluate the integral

$$\int d^3r' \rho(\mathbf{r}') \tag{11}$$

   over all space.

   (b) Evaluate the electric potential of the sphere inside and outside the sphere using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{12}$$

   (Hint: Use the δ-function property to evaluate the r' integral. Choose the observation pont to be on the z axis, which allows the θ' and ϕ' integrals to be evaluated.)

   (c) Evaluate the electric field due to the point charge by finding the gradient of the electric potential you calculated using Eq. (12),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{13}$$

6. (**20 points.**) The electric potential due to an infinitely thin plate (or a large disc of radius R on the xy-plane with $|x|, |y|, |z| \ll R$) with uniform charge density σ is given by the expression

$$\phi(\mathbf{r}) = \frac{\sigma}{2\varepsilon_0}\big[R - |z|\big]. \tag{14}$$

Find the (simplified) expression for the electric field due to the plane by evaluating the gradient of the above electric potential,

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{15}$$

7. (**20 points.**) Consider an electric dipole, with the negative charge $-q$ at the coordinate $(0, 0, -a)$ and the positive charge $+q$ at $(0, 0, a)$, such that the electric dipole moment $\mathbf{p}$ points along the z-axis, $p = 2aq$.

(a) Write the charge density for the electric dipole in terms of δ-functions as

$$\rho(\mathbf{r}) = q\,\delta^{(3)}(\mathbf{r} - a\,\hat{\mathbf{k}}) - q\delta^{(3)}(\mathbf{r} + a\hat{\mathbf{k}}). \tag{16}$$

Integrate the charge density over all space using the property of δ-functions. Interpret your result.

(b) The electric potential due to a charge distribution is given using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{17}$$

Show that the electric potential due to the dipole at the point

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}, \qquad r = \sqrt{x^2 + y^2 + z^2}, \tag{18}$$

is given by the expression

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+a)^2}}\right]. \tag{19}$$

Hint: All the integrals can be completed using the property of δ-functions.

(c) For $a \ll r$ show that the potential is approximately given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{pz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}. \tag{20}$$

(d) Consider the limit when a is made to vanish while q becomes infinite in such a way that $2aq$ remains the finite value p. This is a point dipole. The electric potential for a point dipole is exactly described by Eq. (20). Using polar coordinates write $z = r\cos\theta$ and rewrite the potential of a point dipole in Eq. (20) in the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}. \tag{21}$$

(e) Evaluate the electric field due to a point dipole using

$$\mathbf{E} = -\boldsymbol{\nabla}\phi \tag{22}$$

and express it in the following form,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\big]. \tag{23}$$

Draw the electric field lines of a point dipole for $\mathbf{p} = p\,\hat{\mathbf{z}}$.
Hint: Use $\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}$ and $\boldsymbol{\nabla} r = \hat{\mathbf{r}}$.