Midterm Exam No. 03 (Fall 2020)

PHYS 320: ELECTRICITY AND MAGNETISM I

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Date: 2020 Nov 6

1. (**20 points.**) A point charge q is placed near a perfectly conducting plate.
   (a) Will the charge q experience a force?
   (b) If yes, calculate the force of attraction/repulsion between the charge and conducting plate when the charge is a distance a away from the plate.
   (c) If no, why not?

2. (**20 points.**) A grounded perfectly conducting thin plate is located at $z = 0$ plane. A positive charge q is placed at $\mathbf{r}_1 = a\,\hat{\mathbf{z}}$. A negative charge $-q$ is placed at $\mathbf{r}_2 = 2a\,\hat{\mathbf{z}}$.
   (a) Determine the magnitude and direction of the electrostatic force on the positive charge due to the negative charge.
   (b) Determine the magnitude and direction of the electrostatic force on the positive charge due to the plate. Use method of images.
   (c) Determine the magnitude and direction of the total electrostatic force on the positive charge.

3. (**20 points.**) Consider two grounded, thin, perfect conductors occupying half planes extending radially outward from the z axis. Let these planes intersect at the z axis making an angle of 120° between them. That is, say, the two planes are $\theta = \pi/3$ and $\theta = -\pi/3$. Place a point charge on the plane $\theta = \pi/6$ as described in Figure 1. Determine the resulting image charge configuration, assuming that the method of images extends to these configurations analogous to optical images in a mirror.

4. (**20 points.**) Consider an infinite chain of equidistant alternating point charges $+q$ and $-q$ on the x-axis. Calculate the electric potential at the site of a point charge due to all other charges. This is equal to the work per point charge required to assemble such a cofiguration. In terms of the distance a between neighbouring charges we can derive an expression for this potential to be

$$V = \frac{q}{4\pi\varepsilon_0}\frac{M}{a}, \tag{1}$$

where M is a number defined as the Madelung constant for this hypothetical one-dimensional crystal. Determine M as an infinite sum, and evaluate the sum exactly. (Madelung contants for three-dimensional crystals involve triple sums, which are typically a challenge

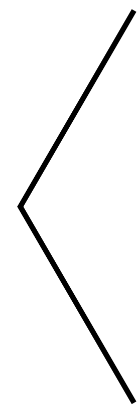

Figure 1: A charge near two intersecting grounded perfect conductors.

to evaluate because of slow convergence.) Next, consider a point charge q placed in between two parallel grounded perfectly conducting plates. Let the plates be positioned at $z = 0$ and $z = a$. For the special situation when the charge q is equidistant from the two plates, find the pattern for the associated infinite image charges. Find the corresponding Madelung constant for this virtual crystal.

5. (**20 points.**) The expression for the electric potential due to a point charge placed in between two parallel grounded perfectly conducting semi-infinite slabs described by

$$\varepsilon(z) = \begin{cases} \infty, & z < 0, \\ \varepsilon_0, & 0 < z < a, \\ \infty, & a < z, \end{cases} \tag{2}$$

is given in terms of the reduced Green's function that satisfies the differential equation $(0 < \{z, z'\} < a)$

$$\left[-\frac{\partial^2}{\partial z^2} + k^2 \right] \varepsilon_0 g(z, z') = \delta(z - z') \tag{3}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$ and $z = a$.

(a) Construct the reduced Green's function in the form

$$g(z, z') = \begin{cases} A \sinh kz + B \cosh kz, & 0 < z < z' < a, \\ C \sinh kz + D \cosh kz, & 0 < z' < z < a, \end{cases} \tag{4}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$g(0, z') = 0, \tag{5a}$$

$$g(a, z') = 0, \tag{5b}$$

$$g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{5c}$$

$$\varepsilon_0 \partial_z g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{5d}$$

(b) After using conditions in Eqs. (5a) and (5b) show that the reduced Green's function can be expressed in the form

$$g(z, z') = \begin{cases} A \sinh kz, & 0 < z < z' < a, \\ C' \sinh k(a - z), & 0 < z' < z < a, \end{cases} \tag{6}$$

where $C' = -C/\cosh ka$. Then, use Eqs. (5c) and (5d) to show that

$$g(z, z') = \begin{cases} \dfrac{1}{\varepsilon_0} \dfrac{\sinh kz \sinh k(a - z')}{k \sinh ka}, & 0 < z < z' < a, \\ \dfrac{1}{\varepsilon_0} \dfrac{\sinh kz' \sinh k(a - z)}{k \sinh ka}, & 0 < z' < z < a. \end{cases} \tag{7}$$

(c) Take the limit $ka \to \infty$ in your solution above, (which corresponds to moving the slab at $z = a$ to infinity,) to obtain the reduced Green's function for a single perfectly conducting slab,

$$\lim_{ka \to \infty} g(z, z') = \frac{1}{\varepsilon_0} \frac{1}{2k} e^{-k|z-z'|} - \frac{1}{\varepsilon_0} \frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{8}$$

This should serve as a check for your solution to the reduced Green's function. Hint: The hyperbolic functions here are defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}). \tag{9}$$