Midterm Exam No. 01 (Fall 2020)

PHYS 320: ELECTRICITY AND MAGNETISM I

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1. (**20 points.**) Using the property of Kronecker δ-function and Levi-Civita symbol evaluate the following using index notation.

$$\delta_{ij}\delta_{jk}\delta_{ki} = \tag{1a}$$

$$\delta_{ij}\varepsilon_{ijk} = \tag{1b}$$

$$\varepsilon_{ijk}\varepsilon_{imn}\varepsilon_{jmn} = \tag{1c}$$

2. (**20 points.**) Given

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} \tag{2}$$

where $\mathbf{B}$ is a constant (homogeneous in space) vector field. Using index notation and the properties of Kronecker δ-function and Levi-Civita symbol in three dimensions expand the left hand side of the vector equation below to express it in the form on the right hand side,

$$\boldsymbol{\nabla} \times \mathbf{A} = \alpha\mathbf{B} + \beta\mathbf{r}. \tag{3}$$

Find the numbers α and β.

3. (**20 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla}(\mathbf{r} \cdot \mathbf{a}) = \alpha\,\mathbf{a} + \beta\,\mathbf{r}, \tag{4}$$

where $\mathbf{a}$ is a constant vector. Thus, find α and β.

4. (**20 points.**) Evaluate the vector area of a spherical ball of radius R using

$$\mathbf{a} = \int_S d\mathbf{a}, \tag{5}$$

where S stands for the surface of the spherical ball. (Caution: The question is discussing the vector area, which is different from the typical surface area of a sphere.)

5. (**20 points.**) Evaluate the integral

$$\int_{-1}^{1} dx\,\delta(1 - 7x)\Big[8x^2 + 2x - 1\Big]. \tag{6}$$

(Caution: Be careful to avoid a possible error in sign.)

6. (**20 points.**) Consider an infinitely long and uniformly charged solid cylinder of radius R with charge per unit length λ.

   (a) Using Gauss's law show that the electric field inside and outside the cylinder is given by

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{\lambda}{2\pi\varepsilon_0} \dfrac{1}{R} \dfrac{r}{R} \hat{\mathbf{r}}, & r < R, \\ \dfrac{\lambda}{2\pi\varepsilon_0} \dfrac{1}{r} \hat{\mathbf{r}}, & r > R, \end{cases} \tag{7}$$

   where $\mathbf{r}$ is now the radial vector transverse to the axis of the cylinder.

   (b) Plot the magnitude of the electric field as a function of r.

   (c) Rewrite your results for the case when the solid cylinder is a perfect conductor?

   (d) Rewrite your results for the case of a uniformly charged hollow cylinder of radius R with charge per unit length λ.

7. (**Optional.**)

   (a) Assume Earth to be a solid spherical ball of uniform density. Consider a hypothetical tunnel passing through the center of Earth and connecting two points on the surface of Earth by a straight line. Determine the time taken, (in minutes) to two siginificant digits, starting from rest, to travel from one point to the other, when a mass is dropped at one end of the tunnel. Ignore friction and the rotational motion of Earth. Use the mass of Earth to be 6.0×10^{24} kg, radius of Earth to be 6.4×10^{6} m. Newton's gravitational constant is $6.67 \times 10^{-11}\,\mathrm{Nm^2/kg^2}$.

   (b) A more realistic density profile of Earth is

$$\rho(r) = \begin{cases} \rho_0, & \text{for} \quad r < \frac{R}{2}, \\ \frac{1}{2}\rho_0, & \text{for} \quad \frac{R}{2} < r < R, \end{cases} \tag{8}$$

   where

$$\rho_0 = \frac{16}{9} \frac{M}{\frac{4\pi}{3} R^3}, \tag{9}$$

   where R is the radius of Earth and M is the mass of Earth. Show that the above density profile leads to the following profile for the gravitational field for Earth,

$$g(r) = \begin{cases} -\dfrac{16}{9} \dfrac{GMr}{R^3}, & \text{for} \quad r < \dfrac{R}{2}, \\ -\dfrac{8}{9} \dfrac{GM}{R^2} \dfrac{1}{2} \left[\dfrac{2r}{R} + \left(\dfrac{R}{2r} \right)^2 \right], & \text{for} \quad \dfrac{R}{2} < r < R, \\ -\dfrac{GM}{r^2}, & \text{for} \quad R < r, \end{cases} \tag{10}$$

where G is Newton's gravitational constant. Plot $g(r)$ as a function of r. Approximate the above gravitational field as

$$g(r) \approx \begin{cases} -\dfrac{GM}{R^2}\dfrac{2r}{R}, & \text{for} \quad r < \dfrac{R}{2}, \\ -\dfrac{GM}{R^2}, & \text{for} \quad \dfrac{R}{2} < r < R, \\ -\dfrac{GM}{r^2}, & \text{for} \quad R < r. \end{cases} \tag{11}$$

Plot the approximate gravitational field and compare it with the exact version. Argue that it is accurate to about ten percent. Determine the new time taken, (in minutes) to two siginificant digits, starting from rest, to travel from one point to the other, when a mass is dropped at one end of the tunnel. Ignore friction and the rotational motion of Earth.

Refer: The gravity tunnel in a non-uniform Earth, by Alexander R. Klotz, Am. J. Phys. 83 (2015) 231; arXiv:1308.1342.