Date: 2014 Dec 2

① For planar geometry with axial symmetry we have

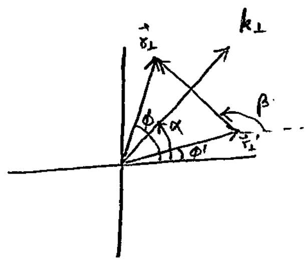
$$G_0(\vec{r},\vec{r}\,') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$= \frac{1}{\epsilon_0}\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp}\,\frac{1}{2k_\perp}\,e^{-k_\perp|z-z'|} \qquad \vec{\rho} = (\vec{r}-\vec{r}\,')_\perp$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^\infty dk_\perp\, J_0(k_\perp \rho)\, e^{-k_\perp|z-z'|}$$

where

$$J_0(k_\perp \rho) = \int_0^{2\pi}\frac{d\beta}{2\pi}\, e^{i k_\perp \rho \cos\beta}$$



② Observe the integral

$$\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\,\frac{e^{ik_z(z-z')}}{k_z^2+k_\perp^2} = \frac{1}{2k_\perp}\,e^{-k_\perp|z-z'|}$$

③ Using ② in the third equality of ① we have

$$G_0(\vec{r},\vec{r}\,') = \frac{1}{2\pi\epsilon_0}\int_0^\infty k_\perp\, dk_\perp\, J_0(k_\perp\rho)\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\,\frac{e^{ik_z(z-z')}}{k_z^2+k_\perp^2}$$

$$= \frac{1}{2\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\, e^{ik_z(z-z')}\underbrace{\boxed{\int_0^\infty k_\perp\, dk_\perp\,\frac{J_0(k_\perp\rho)}{k_\perp^2+k_z^2}}}_{K_0(k_z\rho)}$$

(4) Modified Bessel function of zeroth order is defined as

$$K_0(k_z P) = \int_0^{\infty} k_\perp \, dk_\perp \frac{J_0(k_\perp P)}{k_\perp^2 + k_z^2}$$

or

$$K_0(t) = \int_0^{\infty} s\, ds \frac{J_0(s)}{s^2 + t^2} \qquad 0 < t < \infty$$

(5) Thus, we have the Green's function

$$G_0(\vec{r}, \vec{r}') = \frac{1}{2\pi\epsilon_0} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} K_0(k_z P),$$

which is appropriate for systems with cylindrical symmetry.

(6) We can write

$$G_0(\vec{r}, \vec{r}') = \frac{1}{2\pi\epsilon_0} \int_0^{\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} K_0(k_z P) + \frac{1}{2\pi\epsilon_0} \int_{-\infty}^{0} \frac{dk_z}{2\pi} e^{ik_z(z-z')} K_0(k_z P)$$

$$= \frac{1}{2\pi\epsilon_0} \int_0^{\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} K_0(k_z P) + \frac{1}{2\pi\epsilon_0} \int_0^{\infty} \frac{dk_z}{2\pi} e^{-ik_z(z-z')} \underbrace{K_0(-k_z P)}_{= K_0(k_z P)}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{\pi} \int_0^{\infty} dk_z \left[e^{ik_z(z-z')} + e^{-ik_z(z-z')} \right] K_0(k_z P)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{2}{\pi} \int_0^{\infty} dk_z \cos\left[k_z(z-z')\right] K_0(k_z P)$$

which brings out the oscillatory nature in the z-direction more explicitly.

7 Note that

$$K_0(k_z \rho) = \int_0^\infty k_\perp \, dk_\perp \frac{J_0(k_\perp \rho)}{k_\perp^2 + k_z^2}$$

$$= \int_0^\infty k_\perp \, dk_\perp \int_0^{2\pi} \frac{d\beta}{2\pi} e^{i k_\perp \rho \cos\beta} \frac{1}{k_\perp^2 + k_z^2}$$

$$= 2\pi \int \frac{d^2 k_\perp}{(2\pi)^2} e^{i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp} \frac{1}{k_\perp^2 + k_z^2}.$$

8 The above form suggests

$$\left(-\nabla_\perp^2 + k_z^2\right) \frac{1}{2\pi} K_0(k_z \rho) = \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k_\perp^2 + k_z^2} \left(-\nabla_\perp^2 + k_z^2\right) e^{+i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp}$$

$$= \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{k_\perp^2 + k_z^2} \left(-(+ik_\perp)^2 + k_z^2\right) e^{+i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp}$$

$$= \int \frac{d^2 k_\perp}{(2\pi)^2} e^{i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp}$$

$$= \delta^{(2)}(\vec{r}_\perp - \vec{r}_\perp')$$

Thus, we note that the modified Bessel function satisfies the Green's function

$$\left(-\nabla_\perp^2 + k_z^2\right) \frac{1}{2\pi} K_0(k_z \rho) = \delta^{(2)}(\vec{r}_\perp - \vec{r}_\perp').$$

9 Show that (homework) $\qquad \vec{P} \doteq \vec{r}_\perp - \vec{r}_\perp'$

$$\delta^{(2)}(\vec{r}_\perp - \vec{r}_\perp') \xrightarrow{\vec{r}_\perp \to \vec{r}_\perp'} \frac{\delta(P)}{P}\frac{1}{2\pi}$$

10 Thus we have.

$$\left[-\frac{1}{P}\frac{d}{dP}P\frac{d}{dP} + k_z^2\right]\frac{1}{2\pi}K_0(k_z P) = \frac{\delta(P)}{P}\frac{1}{2\pi}$$

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + 1\right]K_0(t) = \frac{\delta(t)}{t}$$

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + 1\right]K_0(t) = 0 \qquad \text{if} \quad t \neq 0.$$

11 Replacing $t \to it$

$$\left[\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + 1\right]K_0(it) = 0$$

which reveals the relation

$$J_0(t) = K_0(it).$$

(12) Using divergence theorem (in two dimensions) on

$$\left(-\nabla_\perp^2 + k_z^2\right) K_0 = 2\pi\, \delta^{(2)}(\vec{r}_\perp - \vec{r}_\perp{}')$$

we have around $\vec{r}_\perp{}'$

$$-\int_V d^2 r_\perp\, \nabla_\perp^2 K_0 = 2\pi \int d^2 r_\perp\, \delta^{(2)}(\vec{r}_\perp - \vec{r}_\perp{}')$$

$$-\int d\vec{r} \cdot \vec{\nabla}_\perp K_0 = 2\pi$$

$$-2\pi \rho \frac{d}{d\rho} K_0 = 2\pi$$

$$\frac{d}{dt} K_0(t) = -\frac{1}{t}$$

$$\Rightarrow \quad K_0(t) = -\ln t + \text{constant}$$

$$= \ln \frac{2}{t} - \gamma$$

$\gamma = 0.577\ldots$
$= \text{Euler's constant}$