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Scattering formalism

1. For the case: (i) $\rho = 0$, $\vec{J} = 0$
   (ii) $\overleftrightarrow{\mu} = \mu_0 \overleftrightarrow{I}$ (for simplicity)
   (iii) $\frac{\partial}{\partial t} \to -i\omega$

   the Maxwell's equations are.

$$\vec{\nabla} \times \vec{E} = i\omega \vec{B} \qquad \vec{D} = \overleftrightarrow{\epsilon} \cdot \vec{E}$$

$$\vec{\nabla} \times \vec{H} = -i\omega \vec{D} \qquad \vec{B} = \mu_0 \vec{H}.$$

and leads to

$$\left[\vec{\nabla} \times (\vec{\nabla} \times \overleftrightarrow{I}) - \frac{\omega^2}{c^2} \overleftrightarrow{I} \right] \cdot \vec{E}(\vec{r}, \omega) = \frac{\omega^2}{c^2} \left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} - \overleftrightarrow{I} \right) \cdot \vec{E}(\vec{r}, \omega)$$

2. The Green's dyadic is constructed using

$$\left[\vec{\nabla} \times (\vec{\nabla} \times \overleftrightarrow{I}) - \frac{\omega^2}{c^2} \overleftrightarrow{I} \right] \cdot \overleftrightarrow{\Gamma}_0 (\vec{r}, \vec{r}\,'; \omega) = \overleftrightarrow{I} \, \delta^{(3)}(\vec{r} - \vec{r}\,')$$

and has the solution

$$\overleftrightarrow{\Gamma}_0 (\vec{r}, \vec{r}\,'; \omega) = \frac{c^2}{\omega^2} \left(\vec{\nabla} \vec{\nabla} + \frac{\omega^2}{c^2} \overleftrightarrow{I} \right) \frac{e^{-i \frac{\omega}{c} |\vec{r} - \vec{r}\,'|}}{4\pi |\vec{r} - \vec{r}\,'|}$$

3. Using 2 in 1 we have the solution for the electric field.

$$\vec{E}(\vec{r}, \omega) = \vec{E}_0(\vec{r}, \omega) + \left(\vec{\nabla} \vec{\nabla} + \frac{\omega^2}{c^2} \overleftrightarrow{I} \right) \int d^3 r' \, \frac{e^{-i \frac{\omega}{c} |\vec{r} - \vec{r}\,'|}}{4\pi |\vec{r} - \vec{r}\,'|} \left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} - \overleftrightarrow{I} \right) \cdot \vec{E}(\vec{r}, \omega)$$

where.

$$\left[\vec{\nabla} \times (\vec{\nabla} \times \overleftrightarrow{I}) - \frac{\omega^2}{c^2} \overleftrightarrow{I} \right] \cdot \vec{E}_0 (\vec{r}, \omega) = 0.$$

4 In the radiation approximation we have

$$\vec{k} = \frac{\omega}{c}\hat{r}$$

$$k = \frac{\omega}{c}$$

$$|\vec{r} - \vec{r}'| \approx r - \hat{r}\cdot\vec{r}'$$

$$\frac{e^{-i\frac{\omega}{c}|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \approx \frac{e^{-i\frac{\omega}{c}r}}{4\pi r}\, e^{-i\frac{\omega}{c}\hat{r}\cdot\vec{r}'}$$

$$= \frac{e^{-ikr}}{4\pi r}\, e^{-i\vec{k}\cdot\vec{r}'}$$

5 Also, in the radiation approximation,

$$\vec{\nabla}\,\frac{e^{-ikr}}{r} = \left(-ik\hat{r} - \frac{\hat{r}}{r}\right)\frac{e^{-ikr}}{r} \approx -i\vec{k}\,\frac{e^{-ikr}}{r}$$

$$\left(\vec{\nabla}\vec{\nabla} + \frac{\omega^2}{c^2}\overleftrightarrow{1}\right)\frac{e^{-ikr}}{4\pi r} \approx \left(-\vec{k}\vec{k} + \frac{\omega^2}{c^2}\overleftrightarrow{1}\right)\frac{e^{-ikr}}{4\pi r}$$

$$= \frac{e^{-ikr}}{4\pi r}\left(-\vec{k}\,\vec{k}\cdot\overleftrightarrow{1} + k^2\overleftrightarrow{1}\right)$$

$$= -\frac{e^{-ikr}}{4\pi r}\,\vec{k}\times(\vec{k}\times\overleftrightarrow{1})$$

6 Using 4 and 5 in 3 we have.

$$\underset{r\to\infty}{Lt}\ \vec{E}(\vec{r},\omega) = \vec{E}_0(\vec{r},\omega) - \frac{e^{-ikr}}{r}\,\frac{1}{4\pi}\int d^3r'\, e^{-i\vec{k}\cdot\vec{r}'}\,\vec{k}\times(\vec{k}\times\overleftrightarrow{1})\cdot\left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} - \overleftrightarrow{1}\right)\cdot\vec{E}(\vec{r}',\omega)$$

7 We define the scattering amplitude as.

$$\underset{r\to\infty}{Lt}\ \vec{E}(\vec{r},\omega) = \underset{r\to\infty}{Lt}\ \vec{E}_0(\vec{r},\omega) + \vec{F}(\theta,\phi)\ \frac{e^{-ikr}}{r}$$

where the scattering amplitude is read of from 6 as.

$$\vec{F}(\theta,\phi) = -\vec{k}\times(\vec{k}\times\overleftrightarrow{1})\cdot\frac{1}{4\pi}\int d^3r'\ e^{-i\vec{k}\cdot\vec{r}'}\left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0}-\overleftrightarrow{1}\right)\cdot\vec{E}(\vec{r},\omega)$$

→ $\vec{E}(\vec{r},\omega)$ requires solutions near the origin, which requires us to solve. 1.

→ Contributions to $\vec{F}(\theta,\phi)$ in far field gets contributions from the scattering zone when $\left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0}-\overleftrightarrow{1}\right)$ is finite.

8 <u>Rayleigh approximation</u> : (in QM Born-Oppenheimer approximation).

$$\vec{E} = \vec{E}_0(1+z+z^2+\dots) = \frac{\vec{E}_0}{1-z} \qquad z<1.$$

For $\underbrace{k^3\int d^3r'\left\|\frac{\overleftrightarrow{\epsilon}}{\epsilon_0}-\overleftrightarrow{1}\right\|}_{\delta^3} << 1$

δ – scattering length.

$$\vec{F}(\theta,\phi) = -\vec{k}\times(\vec{k}\times\overleftrightarrow{1})\cdot\frac{1}{4\pi}\int d^3r'\ e^{-i\vec{k}\cdot\vec{r}'}\left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0}-\overleftrightarrow{1}\right)\cdot\vec{E}_0(\vec{r},\omega)$$

(9) Let

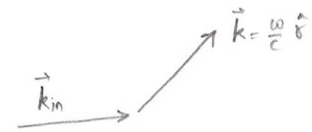
$$\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} - \overleftrightarrow{1} = 4\pi \, \overleftrightarrow{\alpha} \, \delta^{(3)}(\vec{r} - \vec{r}_0)$$

position of the scattering particle

Then, we have.

$$\vec{F}(\theta, \phi) = - \vec{k} \times (\vec{k} \times \overleftrightarrow{1}) \cdot \overleftrightarrow{\alpha} \cdot \vec{E}_0(\vec{r}_0, \omega) \, e^{-i\vec{k}\cdot\vec{r}_0}$$

(10) Further, let



$$\vec{E}_0(\vec{r}, \omega) = \vec{E}_0 \, e^{i\vec{k}_{in}\cdot\vec{r}}$$

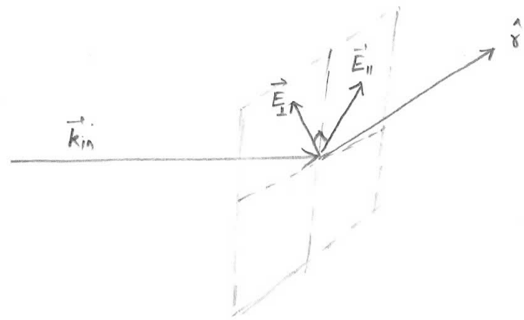
which leads to

$$\vec{F}(\theta, \phi) = - \vec{k} \times (\vec{k} \times \overleftrightarrow{\alpha}) \cdot \vec{E}_0 \, e^{-i(\vec{k} - \vec{k}_{in})\cdot\vec{r}_0}$$

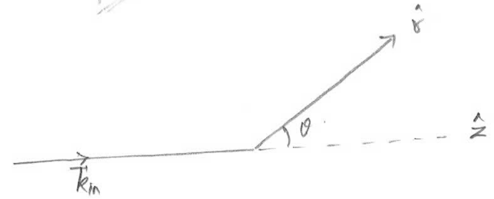
(11) For $\overleftrightarrow{\alpha} = \alpha \overleftrightarrow{1}$

$$\begin{aligned} |\vec{F}(\theta,\phi)|^2 &= \alpha^2 \frac{\omega^4}{c^4} |\hat{r} \times (\hat{r} \times \vec{E}_0)|^2 \\ &= \alpha^2 \frac{\omega^4}{c^4} \left[|\hat{r} \times \vec{E}_0|^2 - \underbrace{|\hat{r} \cdot (\hat{r} \times \vec{E}_0)|^2}_{=0} \right] \\ &= \alpha^2 \frac{\omega^4}{c^4} \left[E_0^2 - (\hat{r} \cdot \vec{E}_0)^2 \right] \end{aligned}$$

(12) $\vec{E}_0 = \vec{E}_{\parallel} + \vec{E}_{\perp}$



→ $\vec{E}_0$ is in the plane perpendicular to $\vec{k}_{in}$



→ $\hat{r} \cdot \vec{E}_0 = E_0 \sin\theta$.

$\vec{E}_0 = E_0 (\cos\phi_0 \,\hat{i} + \sin\phi_0 \,\hat{j})$

$\hat{r} = \sin\theta (\cos\phi \,\hat{i} + \sin\phi \,\hat{j}) + \cos\theta \,\hat{k}$

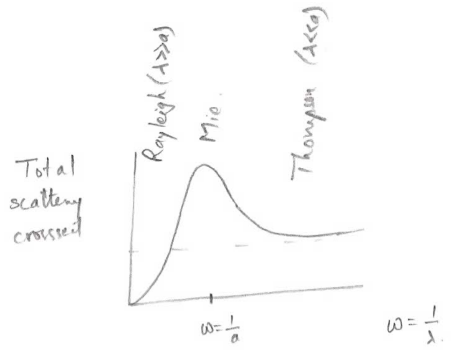
$\hat{r} \cdot \vec{E}_0 = E_0 \sin\theta \cos(\phi - \phi_0)$

$= E_0 \sin\theta$. for $\phi = \phi_0$.

(13) $$|\vec{F}(\theta,\phi)|^2 = \alpha^2 \frac{\omega^4}{c^4} E_0^2 \begin{cases} 1, & \perp r \text{ pol.,} \\ \cos^2\theta, & \parallel \text{ pol.,} \\ \frac{1+\cos^2\theta}{2}, & \text{unpolarized.} \end{cases}$$

→ $|F(\theta,\phi)|^2$ has dimensions of area.

(14)



→ Rayleigh scattering is an elastic collision.

→ Raman scattering is an inelastic collision – some energy is absorbed.