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Green's dyadic

(1)

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{D} = \overleftrightarrow{\epsilon}\cdot\vec{E}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \qquad \vec{B} = \overleftrightarrow{\mu}\cdot\vec{H}$$

(2) Let us analyse the process for a single monochromatic wave.

$$\vec{E}(\vec{r},t) = e^{-i\omega t}\,\vec{E}(\vec{r},\omega)$$

$$\vec{B}(\vec{r},t) = e^{-i\omega t}\,\vec{B}(\vec{r},\omega)$$

This involves

$$\frac{\partial}{\partial t} \rightarrow -i\omega.$$

Further, let

$$\rho = 0 \quad \text{and} \quad \vec{J} = 0$$

And, let $\overleftrightarrow{\mu} = \overleftrightarrow{I}$.

Assumptions:

(i) $\rho = 0$, $\vec{J} = 0$

(ii) $\overleftrightarrow{\mu} = \overleftrightarrow{I}$

(iii) $\frac{\partial}{\partial t} \rightarrow -i\omega$

(3) Thus,

$$\vec{\nabla}\times\vec{E} = i\omega\vec{B} \quad \Rightarrow \quad 0 = \vec{\nabla}\cdot\vec{B}$$

$$\vec{\nabla}\times\vec{H} = -i\omega\vec{D} \quad \Rightarrow \quad 0 = \vec{\nabla}\cdot\vec{D}$$

(4) Since, the bottom equations imply the top ones, we have.

$$\vec{\nabla}\times\vec{E} = i\omega\vec{B}. \qquad \vec{D} = \overleftrightarrow{\epsilon}\cdot\vec{E}$$

$$\vec{\nabla}\times\vec{H} = -i\omega\vec{D}. \qquad \vec{B} = \mu_0\vec{H}$$

(5)
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = i\omega\,\vec{\nabla}\times\vec{B}$$
$$= \mu_0\, i\omega(-i\omega)\,\vec{D}$$
$$= \frac{\omega^2}{c^2}\,\frac{\overleftrightarrow{\epsilon}}{\epsilon_0}\cdot\vec{E}$$
$$\left[\vec{\nabla}\times(\vec{\nabla}\times\overleftrightarrow{1}) - \frac{\omega^2}{c^2}\overleftrightarrow{1}\right]\cdot\vec{E}(\vec{r},\omega) = \frac{\omega^2}{c^2}\left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} - \overleftrightarrow{1}\right)\cdot\vec{E}$$

(6)
$$\left[\vec{\nabla}\times(\vec{\nabla}\times\overleftrightarrow{1}) - \frac{\omega^2}{c^2}\overleftrightarrow{1}\right]\cdot\overleftrightarrow{\Gamma}_0(\vec{r},\vec{r}';\omega) = \overleftrightarrow{1}\cdot\delta^{(3)}(\vec{r}-\vec{r}')$$
$$\left[\vec{\nabla}(\vec{\nabla}\cdot\overleftrightarrow{1}) - \nabla^2\overleftrightarrow{1} - \frac{\omega^2}{c^2}\overleftrightarrow{1}\right]\cdot\overleftrightarrow{\Gamma}_0 = \overleftrightarrow{1}$$
$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right)G_0 = 1$$
$$\left[\vec{\nabla}\vec{\nabla} - \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\overleftrightarrow{1}\right]\cdot\overleftrightarrow{\Gamma}_0 = \overleftrightarrow{1}$$
$$\left[\vec{\nabla}\vec{\nabla} + G_0^{-1}\overleftrightarrow{1}\right]\cdot\overleftrightarrow{\Gamma}_0 = \overleftrightarrow{1}$$
$$\left[G_0\vec{\nabla}\vec{\nabla} + \overleftrightarrow{1}\right]\cdot\overleftrightarrow{\Gamma}_0 = G_0\overleftrightarrow{1}$$
$$\overleftrightarrow{\Gamma}_0 = -G_0\,\vec{\nabla}\,\vec{\nabla}\cdot\overleftrightarrow{\Gamma}_0 + G_0\overleftrightarrow{1}$$

(7)
$$\vec{\nabla}\cdot\left[\vec{\nabla}\times(\vec{\nabla}\times\overleftrightarrow{1}) - \frac{\omega^2}{c^2}\overleftrightarrow{1}\right]\cdot\overleftrightarrow{\Gamma}_0 = \vec{\nabla}\cdot\overleftrightarrow{1}\,\delta^{(3)}$$
$$-\frac{\omega^2}{c^2}\,\vec{\nabla}\cdot\overleftrightarrow{\Gamma}_0 = \vec{\nabla}\,\delta^{(3)}$$

(8) Using (7) in (6)
$$\overleftrightarrow{\Gamma}_0 = +G_0\,\vec{\nabla}\,\frac{c^2}{\omega^2}\,\vec{\nabla}\,\overleftrightarrow{1} + G_0\overleftrightarrow{1}$$ (after integrate by parts).
$$= \frac{c^2}{\omega^2}\left(\vec{\nabla}\vec{\nabla} + \frac{\omega^2}{c^2}\overleftrightarrow{1}\right)G_0$$
$$\overleftrightarrow{\Gamma}_0(\vec{r},\vec{r}';\omega) = \frac{c^2}{\omega^2}\left(\vec{\nabla}\vec{\nabla} + \frac{\omega^2}{c^2}\overleftrightarrow{1}\right)G_0(\vec{r},\vec{r}';\omega)$$
where
$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right)G_0(\vec{r},\vec{r}';\omega) = \delta^{(3)}(\vec{r}-\vec{r}')$$

9 We have the solution

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G_0(\vec{r}-\vec{r}', t-t') = \delta^{(3)}(\vec{r}-\vec{r}')\,\delta(t-t')$$

$$G_0(\vec{r}-\vec{r}', t-t') = \frac{\delta(t-t'-\frac{R}{c})}{4\pi R} \qquad R = |\vec{r}-\vec{r}'|$$

$$= \frac{1}{4\pi R}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega(t-t'-\frac{R}{c})}$$

$$G_0(\vec{r}-\vec{r}', t-t') = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, G_0(\vec{r}-\vec{r}';\omega)$$

$$\Rightarrow \quad G_0(\vec{r}-\vec{r}';\omega) = \frac{e^{-i\frac{\omega}{c}R}}{4\pi R}$$

10 Thus, we have.

$$\left[\vec{\nabla}\times(\vec{\nabla}\times\overleftrightarrow{I}) - \frac{\omega^2}{c^2}\overleftrightarrow{I}\right]\cdot\vec{E}(\vec{r},\omega) = \frac{\omega^2}{c^2}\left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} - \overleftrightarrow{I}\right)\cdot\vec{E}(\vec{r},\omega)$$

$$\vec{E}(\vec{r},\omega) = \vec{E}_0(\vec{r},\omega) + \int d^3r'\; \overleftrightarrow{\Gamma}_0(\vec{r}-\vec{r}';\omega)\cdot\frac{\omega^2}{c^2}\left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} - \overleftrightarrow{I}\right)\cdot\vec{E}(\vec{r}',\omega)$$

where.

$$\left[\vec{\nabla}\times(\vec{\nabla}\times\overleftrightarrow{I}) - \omega^2\,\overleftrightarrow{I}\right]\cdot\vec{E}_0 = 0$$

and

$$\overleftrightarrow{\Gamma}_0(\vec{r}-\vec{r}';\omega) = \frac{c^2}{\omega^2}\left(\vec{\nabla}\vec{\nabla} + \frac{\omega^2}{c^2}\overleftrightarrow{I}\right)\frac{e^{-i\frac{\omega}{c}|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}$$