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Cerenkov radiation

(1) Starting from

$$\frac{\partial U(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot\vec{S}(\vec{r},t) + \vec{J}_e\cdot\vec{E} = 0$$

$E = \int d^3r\, U(\vec{r},t).$

we have shown that

$P = -\frac{dE}{dt}.$

$$-\frac{\partial^2 E}{\partial t\,\partial\Omega} = \frac{1}{c\mu_0}\, r^2\left[c\vec{B}(\vec{r},t)\right]^2$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\left[\hat{r}\times\int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right)\right]^2$$

and

$$-\frac{\partial^2 E}{\partial\omega\,\partial\Omega} = \frac{1}{\pi}\frac{c}{\mu_0}\, r^2\left|\vec{B}(\vec{r},\omega)\right|^2$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\frac{\omega^2}{\pi}\left|\hat{r}\times\vec{J}(\vec{k},\omega)\right|^2$$

(2) Note that

$$(\hat{r}\times\vec{Z})^2 = \epsilon_{ijk}\,\hat{r}_j\, Z_k\,\epsilon_{imn}\,\hat{r}_m\, Z_n.$$

$$= \hat{r}\cdot\hat{r}\;\vec{Z}\cdot\vec{Z} - (\hat{r}\cdot\vec{Z})^2.$$

$$= Z^2 - (\hat{r}\cdot\vec{Z})^2$$

$$|\hat{r}\times\vec{Z}|^2 = |\vec{Z}|^2 - |\hat{r}\cdot\vec{Z}|^2$$

(3) Using (2) in (1) we have.

$$-\frac{\partial^2 E}{\partial t\,\partial\Omega} = \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\left[\int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right)\right]^2$$

$$-\frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\left[\hat{r}\cdot\int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right)\right]^2$$

and

$$-\frac{\partial^2 E}{\partial\omega\,\partial\Omega} = \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\frac{\omega^2}{\pi}\left[\left|\vec{J}(\vec{k},\omega)\right|^2 - \left|\hat{r}\cdot\vec{J}(\vec{k},\omega)\right|^2\right]$$

(4) Note that

$$\vec{\nabla}'\cdot\vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right) = \left\{\vec{\nabla}'\cdot\vec{J}(\vec{r}',t')\right\}_{t'=t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}}$$

$$+\frac{\hat{r}}{c}\cdot\frac{\partial}{\partial t}\vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right)$$

which when integrated over space leads to

$$0 = \int d^3r'\left\{\vec{\nabla}'\cdot\vec{J}(\vec{r}',t')\right\}_{t'=t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}} + \frac{\hat{r}}{c}\cdot\int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right)$$

because $\int d^3r'\,\vec{\nabla}'\cdot\vec{J} = \oint d\vec{a}\cdot\vec{J} = 0$ if $\vec{J}=0$ on surface. Thus,

$$\frac{\hat{r}}{c}\cdot\int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right) = \int d^3r'\left\{\frac{\partial}{\partial t'}\rho(\vec{r}',t')\right\}_{t'=t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}}$$

using $\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$.

5 In the Fourier space local charge conservath read

$$i\vec{k}\cdot\vec{J}(\vec{k},\omega) - i\omega\,\rho(\vec{k},\omega) = 0 \qquad \vec{k} = \frac{\omega}{c}\hat{r}.$$

$$\frac{1}{c}\hat{r}\cdot\vec{J}(\vec{k},\omega) = \rho(\vec{k},\omega)$$

6 Using 4 and 5 in 3 we have.

$$-\frac{\partial^2 E}{\partial t\,\partial\Omega} = \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c}\left[\frac{1}{c^2}\left\{\int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}',t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right)\right\}^2 - \left\{\int d^3r'\,\frac{\partial}{\partial t}\rho\left(\vec{r}',t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right)\right\}^2\right]$$

$$-\frac{\partial^2 E}{\partial\omega\,\partial\Omega} = \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c}\frac{\omega^2}{\pi}\left[\left|\frac{1}{c}\vec{J}(\vec{k},\omega)\right|^2 - \left|\rho(\vec{k},\omega)\right|^2\right]$$

7 Let us introduce two time scales, the macroscopic time T and microscopic time τ.

$$T = \frac{t+t'}{2} \qquad t = T + \frac{\tau}{2}$$

$$\tau = t - t' \qquad t' = T - \frac{\tau}{2}.$$

Further, the Jacobian is:

$$dT\,d\tau = \left\|\begin{matrix} \frac{\partial T}{\partial t} & \frac{\partial T}{\partial t'} \\ \frac{\partial \tau}{\partial t} & \frac{\partial \tau}{\partial t'} \end{matrix}\right\| dt\,dt' = \left\|\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{matrix}\right\| dt\,dt'$$

$$= dt\,dt'.$$

8 Starting from 1 we have.

$$-\frac{\partial^2 E}{\partial\omega\,\partial\Omega} = \frac{1}{4\pi\epsilon_0}\,\frac{1}{4\pi c^3}\,\frac{\omega^2}{\pi}\,\left|\hat{\gamma}\times\vec{J}(\vec{k},\omega)\right|^2$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{4\pi c^3}\,\frac{\omega^2}{\pi}\;\hat{\gamma}\times\left[\int_{-\infty}^{+\infty}dt\; e^{i\omega t}\,\vec{J}(\vec{k},t)\right]^*\cdot\;\hat{\gamma}\times\left[\int_{-\infty}^{+\infty}dt'\; e^{i\omega t'}\,\vec{J}(\vec{k},t')\right]$$

9
$$\int_{-\infty}^{+\infty}dt\; e^{-i\omega t}\,\vec{J}(\vec{k},t)^*\int_{-\infty}^{+\infty}dt'\; e^{i\omega t'}\,\vec{J}(\vec{k},t')$$

$$= \int_{-\infty}^{+\infty}dT\int_{-\infty}^{+\infty}d\tau\; e^{-i\omega\tau}\,\vec{J}(\vec{k},T+\tfrac{\tau}{2})^*\;\vec{J}(\vec{k},T-\tfrac{\tau}{2})$$

10 Substituting 9 in 8 we have.

$$-\frac{\partial^2 E}{\partial\omega\,\partial\Omega} = \frac{1}{4\pi\epsilon_0}\,\frac{1}{4\pi c^3}\,\frac{\omega^2}{\pi}\int_{-\infty}^{+\infty}dT\int_{-\infty}^{+\infty}d\tau\; e^{-i\omega\tau}\;\hat{\gamma}\times\vec{J}(\vec{k},T+\tfrac{\tau}{2})^*\cdot\;\hat{\gamma}\times\vec{J}(\vec{k},T-\tfrac{\tau}{2})$$

which lets us identify the macroscopic power spectrum as.

$$-\frac{\partial^3 E}{\partial T\,\partial\omega\,\partial\Omega} = \frac{1}{4\pi\epsilon_0}\,\frac{1}{4\pi c^3}\,\frac{\omega^2}{\pi}\int_{-\infty}^{+\infty}d\tau\; e^{-i\omega\tau}\;\hat{\gamma}\times\vec{J}(\vec{k},T+\tfrac{\tau}{2})^*\cdot\;\hat{\gamma}\times\vec{J}(\vec{k},T-\tfrac{\tau}{2})$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{4\pi c}\,\frac{\omega^2}{\pi}\int_{-\infty}^{+\infty}d\tau\; e^{-i\omega\tau}\left[\frac{1}{c}\vec{J}(\vec{k},T+\tfrac{\tau}{2})^*\cdot\frac{1}{c}\vec{J}(\vec{k},T-\tfrac{\tau}{2}) - \rho(\vec{k},T+\tfrac{\tau}{2})\;\rho(\vec{k},T-\tfrac{\tau}{2})\right]$$

using 6.

(11) Let us now consider a point charge moving with constant velocity

$$\rho(\vec r', t) = q\, \delta^{(3)}(\vec r' - \vec v t)$$

$$\vec J(\vec r', t) = q\, \vec v\, \delta^{(3)}(\vec r' - \vec v t)$$

(12) Thus, we have.

$$\rho(\vec k, t)^* \rho(\vec k, t') = \left[\int d^3 r'\, e^{-i\vec k\cdot\vec r'} \rho(\vec r', t)\right]^* \int d^3 r''\, e^{-i\vec k\cdot\vec r''} \rho(\vec r'', t')$$

$$= q^2 \left[e^{-i\vec k\cdot\vec v t}\right]^* e^{-i\vec k\cdot\vec v t'}$$

$$= q^2\, e^{i\vec k\cdot\vec v (t-t')}$$

$$= q^2\, e^{i\vec k\cdot\vec v \tau}$$

and

$$\frac{1}{c}\vec J(\vec k, t)^* \cdot \frac{1}{c}\vec J(\vec k, t') = q^2 \frac{v^2}{c^2} e^{i\vec k\cdot\vec v \tau}.$$

(13) Using (12) in (10) we then have.

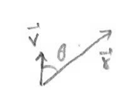
$$-\frac{\partial^3 E}{\partial T\, \partial\omega\, \partial\Omega} = \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c}\frac{q^2\omega^2}{\pi}\left(\frac{v^2}{c^2}-1\right)\int_{-\infty}^{+\infty} d\tau\, e^{-i\omega\tau} e^{i\vec k\cdot\vec v\tau}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c}\frac{q^2\omega^2}{\pi}\left(\frac{v^2}{c^2}-1\right) 2\pi\, \delta(\omega - \vec k\cdot\vec v)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c}\frac{q^2\omega^2}{\pi}\left(\frac{v^2}{c^2}-1\right) 2\pi\, \delta\left(\omega - \omega\frac{v}{c}\cos\theta\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c}\frac{q^2\omega^2}{\pi}\left(\frac{v^2}{c^2}-1\right)$$

$\vec k = \frac{\omega}{c}\hat r$



(14) A charged particle moving with constant speed does not radiate, which is implied by

$$\delta\left(\omega - \omega \frac{v}{c} \cos\theta\right) = 0$$

because

$$\frac{v}{c} \cos\theta < 1 .$$

(15) Now, consider a charged particle moving in a medium described by $\epsilon(\omega)$ and $\mu(\omega)$. Let us define.

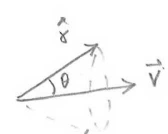
$$n_\epsilon = \sqrt{\frac{\epsilon(\omega)}{\epsilon_0}} \qquad n_\mu = \sqrt{\frac{\mu(\omega)}{\mu_0}} \quad \text{and} \quad n = n_\epsilon n_\mu$$

in terms of which we have.

$$-\frac{\partial^3 E}{\partial T \partial\omega \partial\Omega} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{n_\epsilon^2} \frac{\omega^3 n}{2\pi c} \left(\frac{v^2 n^2}{c^2} - 1\right) \delta\left(\omega - \omega \frac{vn}{c} \cos\theta\right)$$

(16) We observe that we can get a contribution from the δ-function for

$$\frac{vn}{c} \cos\theta = 1$$



cone along which the radiation is observed at a moment.

(17) Using $-\frac{dE}{dt} = P$ we have.

$$\frac{\partial^2 P}{\partial\omega\partial\Omega} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{n_\epsilon^2}\frac{\omega^2 v^2 n^3}{2\pi c^3}\left(1 - \frac{c^2}{v^2 n^2}\right)\delta\left(\omega - \frac{\omega v n}{c}\cos\theta\right)$$

(18) Integrating over angles we have.

$$\frac{\partial P}{\partial\omega} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{n_\epsilon^2}\frac{\omega^2 v^2 n^3}{2\pi c^3}\left(1 - \frac{c^2}{v^2 n^2}\right) 2\pi\int_{-1}^{1} d\cos\theta\;\delta\left(\omega - \frac{\omega v n}{c}\cos\theta\right)$$

$$= \frac{1}{4\pi C_0}\frac{q^2}{n_\epsilon^2}\frac{\omega v n^2}{c^2}\left(1 - \frac{c^2}{v^2 n^2}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q^2\omega v}{c^2}\frac{\mu(\omega)}{\mu_0}\left(1 - \frac{c^2}{v^2}\frac{\epsilon_0\mu_0}{\epsilon(\omega)\mu(\omega)}\right)$$

(19) Energy lost per unit distance traveled is

$$-\frac{\partial^2 E}{\partial t\,\partial\omega} = -\frac{\partial^2 E}{\partial x\,\partial\omega} v$$

Thus,

$$-\frac{\partial^2 E}{\partial x\,\partial\omega} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{c^2}\omega\frac{\mu(\omega)}{\mu_0}\left(1 - \frac{c^2}{v^2}\frac{\epsilon_0\mu_0}{\epsilon(\omega)\mu(\omega)}\right)$$

is the Frank - Tamm formula.

→ linear dependence is ω - dominated by blue color in visible.

→ Cerenkov radiation was predicted by Heaviside in 1889.

20 Total energy lost per unit distance,

$$-\frac{\partial E}{\partial x} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{c^2}\int d\omega \; \omega \; \frac{\mu(\omega)}{\mu_0}\left[1 - \frac{c^2}{v^2}\frac{\epsilon_0\mu_0}{\epsilon(\omega)\mu(\omega)}\right]$$

where the ω-integration extends only over the range.

$$\frac{c^2}{v^2}\frac{1}{\epsilon(\omega)\mu(\omega)} < 1.$$

21 Number of particles emitted is

$$E = N\hbar\omega$$

$$dE = \hbar\omega\, dN$$

Thus,

$$-\frac{\partial^2 E}{\partial x\,\partial\omega} = -\frac{\partial^2 N}{\partial x\,\partial\omega}\hbar\omega.$$

$$-\frac{\partial^2 N}{\partial x\,\partial\omega} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{\hbar c^2}\frac{\mu(\omega)}{\mu_0}\left[1 - \frac{c^2}{v^2}\frac{\epsilon_0\mu_0}{\epsilon(\omega)\mu(\omega)}\right]$$

$$= \frac{\alpha}{c}\frac{\mu(\omega)}{\mu_0}\left[1 - \frac{c^2}{v^2}\frac{\epsilon_0\mu_0}{\epsilon(\omega)\mu(\omega)}\right]$$

$$\alpha = \frac{1}{4\pi\epsilon_0}\frac{q^2}{\hbar c} \simeq \frac{1}{137}$$

$$-\frac{\partial N}{\partial x} = \alpha\int\frac{d\omega}{c}\frac{\mu(\omega)}{\mu_0}\left[1 - \frac{c^2}{v^2}\frac{\epsilon_0\mu_0}{\epsilon(\omega)\mu(\omega)}\right]$$

(22) Estimate

$$-\frac{\partial N}{\partial x} = \alpha \frac{\Delta\omega}{c} \sim \alpha \frac{2\pi}{\lambda}$$

one photon is emitted in a distance of $\frac{137}{2\pi}$ wavelength.

For $\lambda = 500\,nm = 5\times10^{-7}\,m$

$$-\frac{\partial N}{\partial x} = \frac{1 \text{ photon}}{\frac{137}{2\pi}\times 5\times 10^{-7}\,m} = \frac{10^3 \text{ photons}}{cm}$$

(more accurately it is 10^2 photons/cm.)