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Spectral distribution of radiation

1. The detectors are often sensitive to colour (frequency) and one is interested in the spectral distribution of radiation. To determine the spectral distribution, we return back to the Fourier transform in time, to our original equations.

2.
$$\vec{B}(\vec{r},t) = \vec{\nabla} \times \vec{A}(\vec{r},t)$$
$$\vec{E}(\vec{r},t) = -\vec{\nabla}\phi(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}(\vec{r},t)$$
$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\vec{r},t) = \frac{1}{\epsilon_0}\rho(\vec{r},t)$$
$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r},t) = \mu_0 \vec{J}(\vec{r},t)$$

3. We introduced the Green's function
$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{1}{4\pi}G(\vec{r}-\vec{r}', t-t') = \delta^{(3)}(\vec{r}-\vec{r}')\,\delta(t-t')$$
$$G(\vec{r}-\vec{r}', t-t') = \frac{\delta\left(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right)}{|\vec{r}-\vec{r}'|}$$

(4) in terms of which we have the potentials

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \int_{-\infty}^{+\infty} dt' \frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\,\delta\left(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \int_{-\infty}^{+\infty} dt' \frac{\vec{J}(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\,\delta\left(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

(5) Spectral distribution is given in terms of frequency ω, the Fourier transform of time.

$$\phi(\vec{r},\omega) = \int_{-\infty}^{+\infty} dt\; e^{i\omega t}\,\phi(\vec{r},t)$$

$$\vec{A}(\vec{r},\omega) = \int_{-\infty}^{+\infty} dt\; e^{i\omega t}\,\vec{A}(\vec{r},t)$$

(6) In particular

$$\phi(\vec{r},\omega) = \frac{1}{4\pi\epsilon_0}\int d^3r' \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty} dt\; e^{i\omega t}\,\frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\,\delta\left(t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\int_{-\infty}^{+\infty} dt'\; e^{i\omega t'}\,\rho(\vec{r}',t')$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\,\rho(\vec{r}',\omega)$$

7 Similar evaluation can be done for $\vec{A}$ and we have.

$$\phi(\vec{r},\omega) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \, \rho(\vec{r}',\omega)$$

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \, \vec{J}(\vec{r}',\omega)$$

8 In the radiation zone we have

$$|\vec{r}-\vec{r}'| \approx r - \hat{r}\cdot\vec{r}'$$

which inside 7 leads to

$$\phi(\vec{r},\omega) = \frac{1}{4\pi\epsilon_0} \frac{e^{i\frac{\omega}{c}r}}{r} \int d^3r' \, e^{i\frac{\omega}{c}\hat{r}\cdot\vec{r}'} \rho(\vec{r}',\omega)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \rho(\vec{k},\omega)$$

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \vec{J}(\vec{k},\omega)$$

wave vector: $\vec{k} = \frac{\omega}{c}\hat{r}$.

wavelength: $\frac{2\pi}{\lambda} = \frac{\omega}{c}$

9 Next, we find the fields.

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t)$$

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\phi(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}(\vec{r},t)$$

(10)
$$\vec{E}(\vec{r},\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\, \vec{E}(\vec{r},t)$$
$$= \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \left[-\vec{\nabla}\phi(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}(\vec{r},t) \right]$$
$$= -\vec{\nabla}\phi(\vec{r},\omega) + i\omega\, \vec{A}(\vec{r},t)$$

claiming $\vec{A}(\vec{r},t) = 0$ at $t = \pm\infty$.

(11)
$$\vec{\nabla}\phi(\vec{r},\omega) = \frac{1}{4\pi\epsilon_0}\left(\vec{\nabla}\frac{e^{ikr}}{r}\right)\rho(\vec{k},\omega)$$

$\vec{\nabla} e^{ikr} = e^{ikr}\, ik\vec{\nabla}r = ik\hat{r}\, e^{ikr}$

$\vec{\nabla}\frac{1}{r} = -\frac{\hat{r}}{r^2}$.

$$= \frac{1}{4\pi\epsilon_0}\left(ik\frac{\hat{r}}{r} - \frac{\hat{r}}{r^2}\right) e^{ikr}\rho(\vec{k},\omega)$$

next to leading order, and does not contribute in the radiation zone.

$$= \frac{1}{4\pi\epsilon_0}\, i\vec{k}\, \frac{e^{ikr}}{r}\rho(\vec{k},\omega)$$
$$= i\vec{k}\,\phi(\vec{r},\omega)$$

(12) Thus, in the radiation zone we can use.

$$\vec{\nabla} \to i\vec{k}$$
$$\frac{\partial}{\partial t} \to -i\omega$$

which leads to

$$\vec{B}(\vec{r},\omega) = i\vec{k}\times\vec{A}(\vec{k},\omega)$$
$$\vec{E}(\vec{r},\omega) = -i\vec{k}\,\phi(\vec{k},\omega) + i\omega\,\vec{A}(\vec{k},\omega)$$

(13) The expression for the electric field can be further simplified using the local charge conservation equation

$$\frac{\partial}{\partial t} \rho(\vec{r},t) + \vec{\nabla} \cdot \vec{J}(\vec{r},t) = 0$$

$$-i\omega\, \rho(\vec{k},\omega) + i\vec{k} \cdot \vec{J}(\vec{k},\omega) = 0 \qquad \vec{k} = \frac{\omega}{c}\hat{r}$$

$$\rho(\vec{k},\omega) = \frac{1}{\omega} \vec{k} \cdot \vec{J}(\vec{k},\omega)$$

$$= \frac{1}{c} \hat{r} \cdot \vec{J}(\vec{k},\omega)$$

(14) Using (13) in (12) we have.

$$\vec{E}(\vec{r},\omega) = -i\vec{k}\, \phi(\vec{k},\omega) + i\omega \vec{A}(\vec{k},\omega)$$

$$= -i\vec{k} \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \rho(\vec{k},\omega) + i\omega \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \vec{J}(\vec{k},\omega)$$

$$= -i\vec{k} \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \frac{1}{\omega} \vec{k} \cdot \vec{J}(\vec{k},\omega) + i\omega \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \vec{J}(\vec{k},\omega) \qquad \frac{1}{\mu_0\epsilon_0} = c^2.$$

$$= i\omega \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left[\vec{J}(\vec{k},\omega) - \frac{1}{\mu_0\epsilon_0} \frac{\vec{k}}{\omega} \frac{\vec{k}}{\omega} \cdot \vec{J}(\vec{k},\omega) \right] \qquad \vec{k} = \frac{\omega}{c}\hat{r}.$$

$$= -i\omega \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \hat{r} \times \left(\hat{r} \times \vec{J}(\vec{k},\omega) \right)$$

$$= -\hat{r} \times \left(-\hat{r} \times \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (-i\omega) \vec{J}(\vec{k},\omega) \right)$$

$$= -\hat{r} \times c\vec{B}(\vec{r},\omega)$$

(15) Using (14) in (12) we have.

$\vec{k} = \frac{\omega}{c}\hat{r}$.

$$c\vec{B}(\vec{r},\omega) = c\, i\vec{k} \times \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\vec{J}(\vec{k},\omega)$$

$$= -\hat{r} \times \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\vec{J}(\vec{k},\omega)$$

$$\vec{E}(\vec{r},\omega) = -\hat{r} \times c\vec{B}(\vec{r},\omega)$$

$$c\vec{B}(\vec{r},\omega) = \hat{r} \times \vec{E}(\vec{r},\omega)$$

(16) To determine the spectral distribution in the angular distribution of radiation we consider.

$$\frac{\partial}{\partial t}U(\vec{r},t) + \vec{\nabla}\cdot\vec{S}(\vec{r},t) + \underbrace{\vec{J}_e \cdot \vec{E}}_{\vec{J}_e = 0} = 0$$

$\vec{J}_e = 0$ in the radiation zone

$E = \int d^3r\, U$

$$-\frac{\partial E}{\partial t} = \int d^3r\, \vec{\nabla}\cdot\vec{S}(\vec{r},t)$$

$$= \oint d\vec{a}\cdot\vec{S}(\vec{r},t)$$

$$= \int d\Omega\, r^2\, \hat{r}\cdot\vec{S}(\vec{r},t)$$

$\vec{S} = \vec{E}\times\vec{H}$

$= \frac{1}{\mu_0}\vec{E}\times\vec{B}$.

$$-\frac{\partial^2 E}{\partial t\,\partial\Omega} = r^2\,\hat{r}\cdot\vec{S}(\vec{r},t)$$

$$= r^2\,\hat{r}\cdot\vec{E}\times\vec{B}\,\frac{1}{\mu_0}$$

$$= r^2\,\hat{r}\times\vec{E}\cdot\vec{B}$$

$$= r^2\,[c\vec{B}]^2\,\frac{1}{c\mu_0}$$

(17)
$$-\frac{\partial^2 E}{\partial t\, \partial \Omega} = \frac{1}{c\mu_0}\, r^2 \left[c\, \vec{B}(\vec{r},t)\right]^2$$

energy per unit time (power) radiated into solid angle $d\Omega$.

(18) Total energy radiated into solid angle $d\Omega$ is obtained by integrating over time.

$$-\frac{\partial E}{\partial \Omega} = \int_{-\infty}^{+\infty} dt\, -\frac{\partial^2 E}{\partial t\, \partial \Omega}$$

$$= \frac{c}{\mu_0} \int_{-\infty}^{+\infty} dt\; r^2 \left[\vec{B}(\vec{r},t)\right]^2$$

$$= \frac{c}{\mu_0} r^2 \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \vec{B}(\vec{r},\omega) \cdot \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi}\, e^{-i\omega' t}\, \vec{B}(\vec{r},\omega')$$

$$= \frac{c}{\mu_0} r^2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi}\, \vec{B}(\vec{r},\omega) \cdot \vec{B}(\vec{r},\omega') \underbrace{\int_{-\infty}^{+\infty} dt\, e^{-it(\omega+\omega')}}_{2\pi\,\delta(\omega+\omega')}$$

$$= \frac{c}{\mu_0} r^2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \vec{B}(\vec{r},-\omega) \cdot \vec{B}(\vec{r},\omega)$$

$$= \frac{c}{\mu_0} r^2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left|\vec{B}(\vec{r},\omega)\right|^2$$

$$= \frac{1}{\pi} \frac{c}{\mu_0} r^2 \int_0^{\infty} d\omega \left|\vec{B}(\vec{r},\omega)\right|^2.$$

19 Thus, we identify the spectral distribution of the total energy radiated per unit solid angle as

$$- \frac{\partial^2 E}{\partial \omega \, \partial \Omega} = \frac{1}{\pi} \frac{c}{\mu_0} r^2 \left| \vec{B}(\vec{r}, \omega) \right|^2 .$$

$$= \frac{1}{\pi} \frac{c}{\mu_0} \cancel{r^2} \left| - \hat{r} \times \frac{\mu_0}{4\pi} \frac{e^{ikr}}{\cancel{r}} (-i\omega) \vec{J}(\vec{k}, \omega) \right|^2 \frac{1}{c^2} .$$

$$= \frac{1}{\pi} \frac{1}{4\pi \epsilon_0} \frac{\mu_0 \epsilon_0}{4\pi c} \omega^2 \left| \hat{r} \times \vec{J}(\vec{k}, \omega) \right|^2 .$$

$$= \frac{1}{4\pi \epsilon_0} \frac{1}{4\pi c} \frac{1}{\pi} \left| \vec{k} \times \vec{J}(\vec{k}, \omega) \right|^2 .$$

$$= \frac{1}{4\pi \epsilon_0} \frac{1}{4\pi c^3} \frac{\omega^2}{\pi} \left| \hat{r} \times \vec{J}(\vec{k}, \omega) \right|^2 .$$