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Angular distribution of radiated power

(1) The electric and magnetic fields due to radiation is given by

$$c\vec{B}(\vec{r},t) = -\hat{r}\times\frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}',\, t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right)\left[1+O\left(\frac{cr'}{r}\right)\right]$$

$$\vec{E}(\vec{r},t) = -\hat{r}\times c\vec{B}(\vec{r},t)$$

$$c\vec{B}(\vec{r},t) = \hat{r}\times\vec{E}(\vec{r},t)$$

(2) The energy radiated per unit area per unit time is

$$\frac{\partial U}{\partial t}+\vec{\nabla}\cdot\vec{S}=0$$

$$\frac{\text{energy}}{(\text{Area})(\text{time})} = \text{energy flux} = \vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$\frac{\text{energy}}{\text{time}} = P = r^2\int d\Omega\;\hat{r}\cdot\vec{S}$$

$$\frac{dP}{d\Omega} = \frac{1}{\mu_0}\,\hat{r}\cdot\vec{E}\times\vec{B}\;r^2$$

$$= \frac{1}{\mu_0}\,\hat{r}\times\vec{E}\cdot\vec{B}\;r^2$$

$$= \frac{1}{c\mu_0}\,(c\vec{B})^2\;r^2.$$

(3) Using (1) in (2)

$$\frac{dP}{d\Omega} = \frac{1}{c\mu_0}\left[-\hat{r}\times\frac{\mu_0}{4\pi}\frac{1}{\cancel{r}}\int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right)\right]^2 \cancel{r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{4\pi c^3}\left[\hat{r}\times\int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right)\right]^2$$

(4) For a non-relativistic charge particle, $v << c$,

$$\int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}\right) \approx \int d^3r'\,\frac{\partial}{\partial t}\vec{J}\left(\vec{r}', t-\frac{r}{c}\right)$$

$$= \frac{\partial}{\partial t}\int d^3r'\,\vec{J}\left(\vec{r}', t-\frac{r}{c}\right)$$

$\vec{J}(\vec{r}',t) = q\,\vec{V}_q(t)\,\delta^{(3)}(\vec{r}'-\vec{r}_q(t))$

$\vec{r}'$ is bounded by v times T.

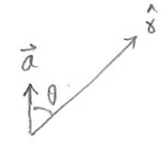
$$= \frac{\partial}{\partial t}\,q\,\vec{V}_q\left(t-\frac{r}{c}\right)$$

$$= q\,\vec{a}_q\left(t-\frac{r}{c}\right)$$

(5) Using (4) in (3)

$$\frac{dP}{d\Omega} = \frac{1}{4\pi\epsilon_0}\,\frac{q^2}{4\pi c^3}\left[\hat{r}\times\vec{a}_q\left(t-\frac{r}{c}\right)\right]^2$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{q^2}{4\pi c^3}\left|\vec{a}_q\left(t-\frac{r}{c}\right)\right|^2 \sin^2\theta$$



$\theta = 0$ – no radiation

$\theta = \frac{\pi}{2}$ – max. rad.

$\theta = \pi$ – no rad.

6 Total power is

$$P = \int d\Omega \, \frac{dP}{d\Omega}$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c^3} \left|\vec{a}_q\left(t - \frac{r}{c}\right)\right|^2 \sin^2\theta$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c^3} \left|\vec{a}_q\left(t - \frac{r}{c}\right)\right|^2 \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} \sin\theta \, d\theta \, \sin^2\theta}_{\int_{-1}^{1} dt\,(1-t^2) = \frac{4}{3}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c^3} \left|\vec{a}_q\left(t - \frac{r}{c}\right)\right|^2 \frac{8\pi}{3}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{2q^2}{3c^3} \left|\vec{a}_q\left(t - \frac{r}{c}\right)\right|^2 \longrightarrow \text{Larmor formula.}$$

7 Let us now improve on the approximation in $v << c$.

$$\vec{K}(\vec{r},t) = \int d^3r' \, \frac{\partial}{\partial t} \vec{J}\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right)$$

$$= \int d^3r' \, \frac{\partial}{\partial t} \left[\vec{J}\left(\vec{r}', t - \frac{r}{c}\right) + \frac{\hat{r}\cdot\vec{r}'}{c} \frac{\partial}{\partial t} \vec{J}\left(\vec{r}', t - \frac{r}{c}\right) + \cdots \right]$$

$$= \frac{\partial}{\partial t} \int d^3r' \, \vec{J}\left(\vec{r}', t - \frac{r}{c}\right) + \frac{\partial^2}{\partial t^2} \int d^3r' \, \frac{\hat{r}\cdot\vec{r}'}{c} \vec{J}\left(\vec{r}', t - \frac{r}{c}\right) + \cdots$$

(8) For a point charge using

$$\vec{J}(\vec{r}', t) = q\, \vec{V}_q(t)\, \delta^{(3)}(\vec{r}' - \vec{r}_q(t))$$

we have.

$$\vec{R}(\vec{r}, t) = \frac{\partial}{\partial t}\, q\, \vec{V}_q\left(t - \frac{r}{c}\right) + \frac{\partial^2}{\partial t^2}\, \frac{1}{c}\, \hat{r} \cdot \vec{r}_q\left(t - \frac{r}{c}\right) \vec{V}_q\left(t - \frac{r}{c}\right) + \cdots$$

$$= \frac{d^2}{dt^2}\, q\, \vec{r}_q(t_e) + \frac{d^2}{dt^2}\, \frac{1}{c}\, \hat{r} \cdot \vec{r}_q(t_e)\, \vec{V}_q(t_e) + \cdots$$

$t_e \equiv t - \frac{r}{c}$.

(9) Remember

charge — $q = \int d^3r'\, \rho(\vec{r}', t)$

electric dipole — $\vec{d}(t) = \int d^3r'\, \vec{r}'\, \rho(\vec{r}', t) = q\, \vec{r}_q(t)$

magnetic dipole — $\vec{\mu}(t) = \frac{1}{2c} \int d^3r'\, \vec{r}' \times \vec{J}(\vec{r}', t) = \frac{q}{2c}\, \vec{r}_q(t) \times \vec{V}_q(t)$

electric quadrapole — $\overleftrightarrow{q}(t) = \int d^3r' \left(3\, \vec{r}'\, \vec{r}' - \overleftrightarrow{1}\, r'^2\right) \rho(\vec{r}', t)$

$$= q\left(3\, \vec{r}_q(t)\, \vec{r}_q(t) - \overleftrightarrow{1}\, r_q(t)^2\right)$$

10 Using 9 in 8

$$\vec{K}(\vec{r},t) = \ddot{\vec{d}} + \frac{d^2}{dt^2}\frac{q}{2c}\Big[\underbrace{\hat{r}\cdot\vec{r}_q(t_e)\,\vec{V}_q(t_e) - \hat{r}\cdot\vec{V}_q(t_e)\,\vec{r}_q(t_e)}_{\hat{r}\times(\vec{V}_q(t_e)\times\vec{r}_q(t_e))}\Big]$$

$$+ \frac{d^2}{dt^2}\frac{q}{2c}\Big[\hat{r}\cdot\vec{r}_q(t_e)\,\vec{V}_q(t_e) + \hat{r}\cdot\vec{V}_q(t_e)\,\vec{r}_q(t_e)\Big]$$

$$= \ddot{\vec{d}} - \hat{r}\times\ddot{\vec{\mu}} + \frac{d}{dt^2}\frac{q}{2c}\,\hat{r}\cdot\Big[\vec{r}_q(t_e)\,\vec{V}_q(t_e) + \vec{V}_q(t_e)\,\vec{r}_q(t_e)\Big]$$

11 Note that

$$\overleftrightarrow{q}(t) = \int d^3r'\,(3\vec{r}'\vec{r}' - \overleftrightarrow{I}r'^2)\,\frac{\partial}{\partial t}\rho(\vec{r}',t) \qquad \left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0\right)$$

$$= -\int d^3r'\,(3\vec{r}'\vec{r}' - \overleftrightarrow{I}r'^2)\,\vec{\nabla}'\cdot\vec{J}(\vec{r}',t)$$

$$= \int d^3r'\,\vec{J}(\vec{r}',t)\cdot\vec{\nabla}'\,(3\vec{r}'\vec{r}' - \overleftrightarrow{I}r'^2)$$

$$= \int d^3r'\,\vec{J}_k\,\nabla'_k\,(3r'_i r'_j - \delta_{ij}r'^2)$$

$$= \int d^3r'\,J_k\,(3\delta_{ik}r'_j + 3r'_i\delta_{kj} - \delta_{ij}\,2r'\,\nabla'_k r')$$

$$= \int d^3r'\,(3J_i r'_j + 3r'_i J_j - 2\delta_{ij}\,r'\,J_k\hat{r}_k)$$

$$= \int d^3r'\,(3\vec{J}\vec{r}' + 3\vec{r}'\vec{J}) - 2\overleftrightarrow{I}\int d^3r'\,r'\,\vec{J}\cdot\hat{r}$$

$$= 3q\,(\vec{r}_q(t)\,\vec{V}_q(t) + \vec{V}_q(t)\,\vec{r}_q(t)) - 2\overleftrightarrow{I}\,r_q(t)\,q\,\vec{V}_q(t)\cdot\hat{r}$$

$$=$$

(12) Using (11) in (10)

$$\vec{K}(\vec{r},t) = \ddot{\vec{d}} - \hat{r}\times\ddot{\vec{\mu}} + \frac{d^2}{dt^2}\frac{1}{2c}\,\hat{r}\cdot\left[\frac{1}{3}\overleftrightarrow{q}(t_e) + \frac{2}{3}\overleftrightarrow{1}\, r_q(t_e)\, \vec{v}_q(t_e)\cdot\hat{r}\right]$$

$$= \ddot{\vec{d}} - \hat{r}\times\ddot{\vec{\mu}} + \frac{1}{2c}\frac{1}{3}\,\hat{r}\cdot\dddot{\overleftrightarrow{q}}(t_e) + \frac{1}{2c}\frac{2}{3}\,\hat{r}\,\frac{d^2}{dt^2}|r_q(t_e)|\,\vec{v}_q(t_e)\cdot\hat{r}$$

(13) Using (12) in (7) and then in (1), noting 4th term does not contribute, $\hat{r}\times\hat{r}=0$,

$$c\vec{B}(\vec{r},t) = -\hat{r}\times\frac{\mu_0}{4\pi}\frac{1}{r}\left[\ddot{\vec{d}} - \hat{r}\times\ddot{\vec{\mu}} + \frac{1}{2c}\frac{1}{3}\,\hat{r}\cdot\dddot{\overleftrightarrow{q}}(t_e)\right]$$

$$\vec{E}(\vec{r},t) = -\hat{r}\times c\vec{B}(\vec{r},t)$$

$$\frac{dP}{d\Omega} = \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\left[\hat{r}\times\left\{\ddot{\vec{d}} - \hat{r}\times\ddot{\vec{\mu}} + \frac{1}{2c}\frac{1}{3}\,\hat{r}\cdot\dddot{\overleftrightarrow{q}}(t_e)\right\}\right]^2$$

(14) If we ignore the contribution from $\overleftrightarrow{q}$ we have.

$$c\vec{B}(\vec{r},t) = -\hat{r}\times\frac{\mu_0}{4\pi}\frac{1}{r}\left[\ddot{\vec{d}} - \hat{r}\times\ddot{\vec{\mu}}\right]$$

$$\vec{E}(\vec{r},t) = +\hat{r}\times\left(\hat{r}\times\frac{\mu_0}{4\pi}\frac{1}{r}\left[\ddot{\vec{d}} - \hat{r}\times\ddot{\vec{\mu}}\right]\right)$$

$$= \hat{r}\times\frac{\mu_0}{4\pi}\frac{1}{r}\left[\hat{r}\times\ddot{\vec{d}} - \hat{r}\times(\hat{r}\times\ddot{\vec{\mu}})\right]$$

$$= \hat{r}\times\frac{\mu_0}{4\pi}\frac{1}{r}\left[\hat{r}\times\ddot{\vec{d}} - \hat{r}\,\underbrace{\ddot{\vec{\mu}}\cdot\hat{r}}_{=0} + \ddot{\vec{\mu}}\right] \quad \hat{r}\times\hat{r}=0.$$

$$= \hat{r}\times\frac{\mu_0}{4\pi}\frac{1}{r}\left[\hat{r}\times\ddot{\vec{d}} + \ddot{\vec{\mu}}\right]$$

15 Note that 14 is invariant under dual transformation

$$\vec{E} \to c\vec{B} \qquad \text{and} \qquad \vec{d} \to \vec{\mu}$$
$$c\vec{B} \to -\vec{E} \qquad\qquad\qquad \vec{\mu} \to -\vec{d}.$$

16 Further

$$\frac{dP}{d\Omega} = \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\left[\hat{r}\times\left(\ddot{\vec{d}} - \hat{r}\times\ddot{\vec{\mu}}\right)\right]^2$$
$$= \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\left[\left(\hat{r}\times\ddot{\vec{d}}\right)^2 + \left(\hat{r}\times\ddot{\vec{\mu}}\right)^2 + 2\,\hat{r}\cdot\left(\ddot{\vec{d}}\times\ddot{\vec{\mu}}\right)\right]$$

using

$$\left(\hat{r}\times\vec{Z}\right)^2 = \epsilon_{ijk}\,\hat{r}_j Z_k\,\epsilon_{imn}\,\hat{r}_m Z_n.$$
$$= Z^2 - (\hat{r}\cdot\vec{Z})^2$$

Note that the interference term here does not contribute to total power P because.

$$\int_0^\pi \sin\theta\, d\theta\, \cos\theta = 0.$$

Thus,

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3c^3}\left[\left(\ddot{\vec{d}}\right)^2 + \left(\ddot{\vec{\mu}}\right)^2\right]$$